FUNCTIONAL ANALYSIS
PRODUCT SPACES

Suppose we have \( \{X_\alpha\}_{\alpha \in A} \).

We define a new space called the product space.

\[
x \in \prod_{\alpha \in A} X_\alpha \text{ if } x = \{x_\alpha\}_{\alpha \in A} \quad x_\alpha \in X_\alpha
\]

Alternatively we could define \( \prod_{\alpha \in A} X_\alpha \) as the set of all functions whose domain is \( A \) and such that \( f(\alpha) \in X_\alpha \).

We can define a topology as follows.

Let \( G = \{x \in X : x_{\alpha_i} \in U_{\alpha_i}, i = 1, 2, \ldots, n \text{ for some } n\} \)

\( U_{\alpha_i} \) open in \( X_{\alpha_i} \)

We take all such sets \( G \) as a basis for the topology in \( X \).

A directed set \( \{X_\beta\} \) is a set in which

(i) there is a partial ordering of the indices \( \beta \),
(ii) given \( \beta_1 \beta_2 \exists \beta_3 \) such that \( \beta_3 > \beta_1, \beta_3 > \beta_2 \).

A directed set \( \{x_\beta\} \) is said to converge to a limit point \( x \) if, given any neighbourhood \( U \) of \( x \exists \) an index \( \beta \) such that \( x_\gamma \in U \) whenever \( \gamma > \beta \).

Convergence in a product space is co-ordinate-wise convergence i.e.

\( \{x^\beta\} \to x \in X \iff x^\beta_\alpha \to x_\alpha \in X_\alpha \text{ for every } \alpha \)

**Vector Spaces** \( E \) is called a vector space over the field \( F \) if \( E \) is a set with two operations of addition and scalar multiplication, the first mapping \( E \times E \) to \( E \) and the second mapping \( F \times E \) to \( E \) in such a way that the following conditions hold.

(i) \( E \) is an Abelian group under addition,
(ii) \( \alpha(x + y) = \alpha x + \alpha y \),
(iii) \( (\alpha + \beta)x = \alpha x + \beta x \),
(vi) \( \alpha(\beta x) = (\alpha \beta)x \),
(v) \( 1.x = x \).
**Normed Vector Spaces** Let $E$ be a vector space over the real or complex numbers. $E$ is a normed space if every $x \in V$ is associated with a non-negative real number $\|x\|$ which has the properties:

(i) $\|x\| = 0 \iff x = 0$,
(ii) $\|\lambda x\| = |\lambda| \|x\|$,
(iii) $\|x + y\| = \|x\| + \|y\|$. 

We can always define a metric on a normed vector space by $p(x, y) = \|x - y\|$ but the converse is not necessarily true.

**Example** Consider $\ell^\infty$, the space of all bounded sequences $\{x_i\}$. Define $p(x, y) = \sum |x_i - y_i|$. Then this is a metric.

In a topological space a set $B$ is said to be bounded if, given any neighbourhood $U$ of the origin, for some $n$, $nU \supset B$.

In a normed vector space there are always bounded neighbourhood of the origin e.g. $B_1 = \{x : \|x\| \leq 1\}$

In $\ell^\infty$ a basic neighbourhood $U$ is defined as follows. Let $\varepsilon > 0$. Let $n_1n_2\ldots n_k$ be a finite sequence of integers let $U = \{\{x_n\} : |x_{n_i}| < \varepsilon i = 1, 2, \ldots, k\}$.

Now choose $m \notin \{n_1n_2\ldots n_k\}$ and let $V = \{\{x_n\} : |x_m| < 1\}$.

Then $NV \supset U$ for any $N$ i.e. no neighbourhood of the origin is bounded.

**A Banach Space** is a normed vector space which is complete for the metric defined by the norm.

The classical examples of Banach spaces are

- $\ell^p$ - the space of all sequences $\{x_n\}$ such that $\sum_{n=1}^\infty |x_n|^p < \infty$.
- $L^p(0, 1)$ - the space of all measurable functions $f(x)$ such that $\int_0^1 |f(x)|^p dx < \infty$.

These are examples of more general spaces $L^p(X, \mu)$.

The norms usually defined on them are

- $\ell^p: \|x\| = \left(\sum |x^n|^p\right)^{\frac{1}{p}}$
- $L^p: \|f\| = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}$ where $f = g$ means $f \equiv g$ p.p.
Lemma If \( \alpha \) and \( \beta \) are non-negative real numbers and if \( 0 < \lambda < 1 \) then
\[
\alpha^\lambda \beta^{1-\lambda} \leq \lambda \alpha + (1-\lambda)\beta \quad \text{with equality} \iff \alpha = \beta .
\]

Proof Let \( \phi(t) = (1-\lambda) + \lambda t - t^\lambda \) then \( \phi'(t) = \lambda (1-t^{\lambda-1}) \) therefore \( \phi'(t) < 0 \)
if \( t < 1 \), \( = 0 \) if \( t = 1 \) and \( > 0 \) if \( t > 1 \). Therefore \( \phi(1) \leq \phi(t) \) for all \( t > 0 \) with equality \( \iff t = 1 \)
\[
(1-\lambda) + \lambda t = t^\lambda \geq 0 \quad \text{with equality} \iff t = 1.
\]
\[
(1-\lambda) + \lambda \frac{a}{\beta} - \frac{a^\lambda}{\beta} \geq 0 \quad \text{with equality} \iff \alpha = \beta
\]
i.e. \( \alpha^\lambda \beta^{1-\lambda} \leq \lambda \alpha + (1-\lambda)\beta \).

Holder’s Inequality let \( p > 2 \) and let \( \frac{1}{p} + \frac{1}{q} = 1 \)

(i) for any \( \{x_n\} \in \ell^p \) and \( \{y_n\} \in \ell^q \)
\[
\sum |x_n y_n| \leq \left( \sum |x_n|^p \right)^{\frac{1}{p}} \left( \sum |y_n|^q \right)^{\frac{1}{q}}
\]
(ii) for any \( f \in L^p, g \in L^q \)
\[
\int |fg| \, dx \leq \left( \int |f|^p \, dx \right)^{\frac{1}{p}} \left( \int |g|^q \, dx \right)^{\frac{1}{q}}
\]

Proof (i) First suppose that \( \sum |x_n|^p = 1 \quad \sum |y_n|^q = 1 \). By the Lemma, taking \( \lambda = \frac{1}{p}, \quad 1 - \lambda = \frac{1}{q}, \)
\[
|X_n y_n| \leq \lambda |x_n|^p + (1 - \lambda) |y_n|^q
\]
therefore \( \sum |x_n y_n| \leq 1 \).
Now consider the sequences
\[
\xi_n = \frac{x_n}{(\sum |x_n|^p)^{\frac{1}{p}}} \quad \eta_n = \frac{y_n}{(\sum |y_n|^q)^{\frac{1}{q}}}
\]
Then \( \sum |\xi_n|^p = 1 \quad \sum |\eta_n|^q = 1 \) therefore \( \sum |\xi_n \eta_n| \leq 1 \). Hence the result.

(ii) Proved in a similar way.

Minkowski’s Inequality (i) Let \( p > 1 \). If \( \{x_n\} \in \ell^p \{y_n\} \in \ell^p \) then \( \{x_n + y_n\} \in \ell^p \) and
\[
\left( \sum |x_n + y_n|^p \right)^{\frac{1}{p}} \leq \left( \sum |x_n|^p \right)^{\frac{1}{p}} + \left( \sum |y_n|^p \right)^{\frac{1}{p}}
\]
(ii) If \( f, g \in L^p \) then \( f + g \in L^p \) and

\[
\left( \int |f + g|^p \, dx \right)^{\frac{1}{p}} \leq \left( \int |f|^p \, dx \right)^{\frac{1}{p}} + \left( \int |g|^p \, dx \right)^{\frac{1}{p}}
\]

Proof (i) For any \( N \)

\[
\sum_{1}^{N} |x_n + y_n|^p \leq \sum_{1}^{N} |x_n + y_n|^{p-1}|x_n| + \sum_{1}^{N} |x_n + y_n|^{p-1}|y_n| \\
\leq \left( \sum_{1}^{N} |x_n|^p \right)^{\frac{1}{p}} \left( \sum_{1}^{N} |x_n + y|^{q(p-1)} \right)^{\frac{1}{q}} \\
+ \left( \sum_{1}^{N} |y_n|^p \right)^{\frac{1}{p}} \left( \sum_{1}^{N} |x_n + y_n|^{q(p-1)} \right)^{\frac{1}{q}} \\
= \left( \sum_{1}^{N} |x_n + y_n|^p \right)^{\frac{1}{p}} \left[ \left( \sum_{1}^{N} |x_n|^p \right)^{\frac{1}{p}} + \left( \sum_{1}^{N} |y_n|^p \right)^{\frac{1}{p}} \right]
\]

\[
\left( \sum_{1}^{N} |x_n + y_n|^p \right)^{\frac{1}{p}} \leq \left( \sum_{1}^{N} |x_n|^p \right)^{\frac{1}{p}} + \left( \sum_{1}^{N} |y_n|^p \right)^{\frac{1}{p}} \\
\leq \left( \sum_{1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} + \left( \sum_{1}^{\infty} |y_n|^p \right)^{\frac{1}{p}}
\]

Hence the result.

(ii) Similar proof.

Using the above results it is easy to verify that \( \ell^p \) and \( L^p \) are normed vector spaces.

\[
\ell^1 : \{ \{ x_n \} : \sum |x_n| < \infty \} \\
\| \{ x_n \} \| = \sum |x_n|
\]

\[
L^1(0 \ 1) : \{ f(x) : \int_{0}^{1} |f(x)| \, dx < \infty \} \\
\| f \| = \int_{0}^{1} |f(x)| \, dx
\]

\[
\ell^\infty : \{ \{ x_n \} : \{ x_n \} \text{bounded} \} \\
\| \{ x_n \} \| = \sup \{|x_n|\}
\]

\[
L^\infty(0 \ 1) : \{ f(x) : |f(x)| < M \ p.p. \} \\
\| f \| = \sup \{ M : |f(x)| < M \ p.p. \}
\]

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Lemma The normed vector space $E$ is complete $\Leftrightarrow \sum_{i=1}^{\infty} \xi_n$ exists in $E$ whenever $\{\xi_n\}$ is a sequence of vectors in $E$ such that $\sum \|\xi_n\| < \infty$.

Proof (i) Let $\eta_n = \sum_{r=1}^{n} \xi_r$

$$\|\eta_n - \eta_m\| = \left\| \sum_{r=m+1}^{n} \xi_r \right\| \leq \sum_{r=m+1}^{n} \|\xi_r\| < \varepsilon$$

If $m$ is sufficiently large. Therefore $\{\eta_n\}$ is a cauchy sequence therefore $\sum_{r=1}^{\infty} \xi_n$ exists.

(ii) Let $\{\zeta_n\}$ be a cauchy sequence in $E$. We can find $\{n_r\}$ such that $\|\zeta_{n_{r+1}} - \zeta_{n_r}\| \leq \frac{1}{2^r}$

Write $\xi_1 = \zeta_{n_1}$ $\xi_r = \zeta_{n_{r+1}} - \zeta_{n_r}$

$\sum \|\xi_r\| < \infty$ therefore $\sum_{r=1}^{n} \xi_r \rightarrow \zeta$ as $n \rightarrow \infty$ i.e. $\zeta_{n_r} \rightarrow \zeta$ as $r \rightarrow \infty$ therefore $\zeta_n \rightarrow \zeta$ as $n \rightarrow \infty$.

Theorem (Riesz-Fischer) The $L^p$ spaces are complete.

Proof Let $\{f_n\}$ be a sequence in $L^p$ such that $\sum_{r=1}^{\infty} \|F_n\| = m < \infty$.

Put $g_n(x) = \sum_{r=1}^{n} |f_r(x)|$.

At every point $x$ the increasing sequence $g_n(x)$ has a limit (finite or infinite). Denote this limit by $g(x)$ then $g(x)$ is measurable and

$$\int |g(x)|^p \, dx = \lim_{n \rightarrow \infty} \int |g_n(x)|^p \, dx \leq m^p$$

therefore $g(x) \in L^p$ and $g(x) < \infty$ p.p. therefore $\sum_{r=1}^{\infty} f_n(x)$ is absolutely convergent p.p. to some function $f(x)$.

$$\left| f(x) - \sum_{r=1}^{n} f_r(x) \right|^p \leq \left( \sum_{r=1}^{\infty} |f_r(x)| \right)^p \leq (g(x))^p$$

Since $(g(x)) \in L^p$

$$\lim \int \left( \sum_{r=1}^{\infty} |f_r(x)| \right)^p \, dx = \int \lim \sum_{r=1}^{\infty} |f_r(x)|^p \, dx = 0$$

by Lebesgues’s Dominated Convergence theorem therefore

$\int |f(x) - \sum_{r=1}^{n} f_r(x)|^p \rightarrow 0$ as $n \rightarrow \infty$ therefore

$||f(x) - \sum_{r=1}^{n} f_r(x)|| \rightarrow 0$ as $n \rightarrow \infty$.

Hence by the Lemma $L^p$ is complete.
Other examples of Banach Spaces (i) The set $C(X)$ of all real-valued, or complex valued, functions defined and continuous on a complex space $X$ where $\|F\| = \sup\{|f(x)| : x \in X\}$.

Convergence in this norm is uniform.

(ii) Set of all functions $f(z)$ analytic on the unit disc with

$$\|F\| = \sup\{|f(z)| : |z| \leq 1\}$$

(iii) Set of all functions $f(z)$ analytic on the unit disc with

$$\|F\| = \iint_{|z| \leq 1} |f| dxdy$$

(iv) The set of all functions $f(z)$ harmonic on the unit circle with

$$\|f\| = \sup\{|f(z)| : |z| \leq 1\}$$

$f(z)$ is harmonic if $\frac{1}{\pi} \oint_C f(z) \, dz = f(z_0) \quad C = \{z : |z - z_0| = \frac{\ell}{2\pi}\}$. 