FUNCTIONAL ANALYSIS
PRODUCT SPACES

Suppose we have $\{X_\alpha\}_{\alpha\in A}$.

We define a new space called the product space.

$$x \in \prod_{\alpha\in A} X_\alpha \text{if} x = \{x_\alpha\}_{\alpha\in A}\ x_\alpha \in X_\alpha$$

Alternatively we could define $\prod_{\alpha\in A} X_\alpha$ as the set of all functions whose domain is A and such that $f(\alpha) \in X_\alpha$.

We can define a topology as follows.

Let $G = \{x \in X : x_{\alpha_i} \in U_{\alpha_i}\ i = 1, 2, \ldots, n$ for some $n\}$

U_{α_i} open in X_{α_i}

We take all such sets G as a basis for the topology in X.

A directed set $\{X_\beta\}$ is a set in which

(i) there is a partial ordering of the indices β,

(ii) given $\beta_1\beta_2\exists\beta_3$ such that $\beta_3 > \beta_1,\ \beta_3 > \beta_2$.

A directed set $\{x_\beta\}$ is said to converge to a limit point x if, given any neighbourhood U of $x\exists$ an index β such that $x_\gamma \in U$ whenever $\gamma > \beta$.

Convergence in a product space is co-ordinate-wise convergence i.e.

$$\{x^\beta\} \to x \in X \Leftrightarrow x^\beta_\alpha \to x_\alpha \in X_\alpha \text{ for every } \alpha$$

Vector Spaces E is called a vector space over the field F if E is a set with two operations of addition and scalar multiplication, the first mapping $E \times E$ to E and the second mapping $F \times E$ to E in such a way that the following conditions hold.

(i) E is an Abelian group under addition,

(ii) $\alpha(x + y) = \alpha x + \alpha y$,

(iii) $(\alpha + \beta)x = \alpha x + \beta x$,

(vi) $\alpha(\beta x) = (\alpha\beta)x$,

(v) $1.x = x$.

Normed Vector Spaces Let E be a vector space over the real or complex numbers. E is a normed space if every $x \in V$ is associated with a non-negative real number $\|x\|$ which has the properties:

(i) $\|X\| = 0 \Leftrightarrow x = 0$,

(ii) $\|\lambda x\| = |\lambda| \|X\|$,

(iii) $\|x + y\| = \|x\| + \|Y\|$.

We can always define a metric on a normed vector space by $p(x, y) = \|x - y\|$ but the converse is not necessarily true.

Example Consider ℓ^∞, the space of all bounded sequences $\{x_i\}$. Define $p(x, y) = \sum \frac{|y_i - x_i|}{2^4}$. Then this is a metric.

in a topological space a set B is said to be bounded if, given any neighbourhood U of the origin, for some n, $nU \supset B$.

In a normed vector space there are always bounded neighbourhood of the origin e'g' $B_1 = \{x : \|x\| \leq 1\}$

In ℓ^∞ a basic neighbourhood U is defined as follows. Let $\varepsilon > 0$. Let $n_1 n_2 \ldots n_k$ be a finite sequence of integers let $U = \{\{x_n\} : |x_{n_i}| < \varepsilon \ i = 1, 2, \ldots, k\}$.

Now choose $m \notin \{n_1 n_2 \ldots n_k\}$ and let $V = \{\{x_n\} : |x_m| < 1\}$.

Then $NV \supset U$ for any N i.e. no neighbourhood of the origin is bounded.

A Banach Space is a normed vector space which is complete for the metric defined by the norm.

The classical examples of Banach spaces are

ℓ^P- the space of all sequences $\{x_n\}$ such that $\sum_{n=1}^\infty |x_n|^p < \infty$.

$L^P(0, 1)$- the space of all measurable functions $f(x)$ such that $\int_0^1 |f(x)|^p \, dx < \infty$.

These are examples of more general spaces $L^P(X, \mu)$.

The norms usually defined on them are

$\ell^P : \|x\| = \left(\sum |x^n|^p\right)^{\frac{1}{p}}$

$L^P : \|f\| = \left(\int_0^1 |f(x)|^P \, dx\right)^{\frac{1}{p}}$ where $f = g$ means $f \equiv g$ p.p.

Lemma If α and β are non-negative real numbers and if $0 < \lambda < 1$ then $\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$ with equality $\Leftrightarrow \alpha = \beta$.

Proof Let $\phi(t) = (1-\lambda) + \lambda t - t^\lambda$ then $\phi'(t) = \lambda(1 - t^{\lambda-1})$ therefore $\phi'(t) < 0$ if $t < 1$, $= 0$ if $t = 1$ and > 0 if $t > 1$. Therefore $\phi(1) \leq \phi(t)$ for all $t > 0$ with equality $\Leftrightarrow t = 1$

$(1-\lambda) + \lambda t = t^\lambda \geq 0$ with equality $\Leftrightarrow t = 1$.

$(1-\lambda) + \lambda\frac{\alpha}{\beta} - \frac{\alpha^\lambda}{\beta^\lambda} \geq 0$ with equality $\Leftrightarrow \alpha = \beta$

i.e. $\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$.

Holder's Inequality let $p > 2$ and let $\frac{1}{p} + \frac{1}{q} = 1$

(i) for any $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^q$

$$\sum |x_n y_n| \leq \left(\sum |x_n|^p\right)^{\frac{1}{p}} \left(\sum |y_n|^q\right)^{\frac{1}{q}}$$

(ii) for any $f \in L^P$ $g \in L^q$

$$\int |fg|\, dx \leq \left(\int |f|^p\, dx\right)^{\frac{1}{p}} \left(\int |g|^q\, dx\right)^{\frac{1}{q}}$$

Proof (i) First suppose that $\sum |x_n|^p = 1 \quad \sum |y_n|^q = 1$. By the Lemma, taking $\lambda = \frac{1}{p}$, $1 - \lambda = \frac{1}{q}$,

$$|X_n y|n| \leq \lambda |x_n|^P + (1-\lambda)|y_n|^q$$

therefore $\sum |x_n y_n| \leq 1$.

Now consider the sequences

$$\xi_n = \frac{x_n}{(\sum |x_n|^p)^{\frac{1}{p}}} \quad \eta_n = \frac{y_n}{(\sum |y_n|^q)^{\frac{1}{q}}}$$

Then $\sum |\xi_n|^p = 1 \quad \sum |\eta_n|^q = 1$ therefore $\sum |\xi_n \eta_n| \leq 1$. Hence the result.

(ii) Proved in a similar way.

Minkowski's Inequality (i) Let $p > 1$. If $\{x_n\} \in \ell^p \{y_n\} \in \ell^p$ then $\{x_n + y_n\} \in \ell^P$ and

$$\left(\sum |x_n + y_n|^P\right)^{\frac{1}{p}} \leq \left(\sum |x_n|^p\right)^{\frac{1}{p}} + \left(\sum |y_n|^p\right)^{\frac{1}{p}}$$

(ii) If $f, g \in L^P$ then $f+g \in L^p$ and

$$\left(\int |f+g|^p \, dx\right)^{\frac{1}{p}} \leq \left(\int |f|^p \, dx\right)^{\frac{1}{p}} + \left(\int |g|^p \, dx\right)^{\frac{1}{p}}$$

Proof (i) For any N

$$\begin{aligned}
\sum_1^N |x_n + y_n|^p &\leq \sum_1^N |x_n + y_n|^{p-1}|x_n| + \sum_1^N |x_n + y_n|^{p-1}|y_n| \\
&\leq \left(\sum_1^N |x_n|^p\right)^{\frac{1}{p}} \left(\sum_1^N |x_n + y)n|^{q(p-1)}\right)^{\frac{1}{q}} \\
&\quad + \left(\sum_1^N |x_n|^p\right)^{\frac{1}{p}} \left(\sum_1^N |x_n + y_n|^{q(p-1)}\right)^{\frac{1}{q}} \\
&= \left(\sum |x_n + y_n|^p\right)^{\frac{1}{q}} \left[\left(\sum_1^N |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_1^N |y_n|^p\right)^{\frac{1}{p}}\right] \\
\left(\sum_1^N |x_n + y_n|^P\right)^{\frac{1}{p}} &\leq \left(\sum_1^N |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_1^N |y_n|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_1^\infty |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_1^\infty |y_n|^p\right)^{\frac{1}{p}}
\end{aligned}$$

Hence the result.

(ii) Similar proof.

Using the above results it is easy to verify that ℓ^p and L^p are normed vector spaces.

$$\begin{aligned}
\ell^1 &: \{\{x_n\} : \sum |x_n| < \infty\} \\
&\quad \|\{x_n\}\| = \sum |x_n| \\
L^1(0\ 1) &: \{f(x) : \int_0^1 |f(x)| \, dx < \infty\} \\
&\quad \|f\| = \int_0^1 |f(x)| \, dx \\
\ell^\infty &: \{\{x_n\} : \{x_n\}\text{bounded}\} \\
&\quad \|\{x_n\}\| = \sup\{|x_n|\} \\
L^\infty(0\ 1) &: \{f(x)\ |f(x)| < M\ p.p\} \\
&\quad \|f\| = \sup\{M : |f(x)| < M\ p.p.\}
\end{aligned}$$

Lemma The normed vector space E is complete $\Leftrightarrow \sum_1^\infty \xi_n$ exists in E whenever $\{\xi_n\}$ is a sequence of vectors in E such that $\sum \|\xi_n\| < \infty$.

Proof (i) Let $\eta_n = \sum_{r=1}^n \xi_r$

$$\|\eta_n - \eta_m\| = \left\| \sum_{r=m+1}^n \xi_r \right\| \leq \sum_{r=m+1}^n \|\xi_r\| < \varepsilon$$

If m is sufficiently large. Therefore $\{\eta_n\}$ is a cauchy sequence therefore $\sum_1^\infty \xi_n$ exists.

(ii) Let $\{\zeta_n\}$ be a cauchy sequence in E. We can find $\{n_r\}$ such that $\|\zeta_{n_{r+1}} - \zeta_{n_r}\| \leq \frac{1}{2}r$
Write $\xi_1 = \zeta_{n_1}$ $\xi_r = \zeta_{n_{r+1}} - \zeta_{n_r}$
$\sum \|\xi_r\| < \infty$ therefore $\sum_{r=1}^n \xi_r \to \zeta$ as $n \to \infty$ i.e. $\zeta_{n_r} \to \zeta$ as $r \to \infty$ therefore $\zeta_n \to \zeta$ as $n \to \infty$.

Theorem (Riesz-Fischer) The L^p spaces are complete.

Proof Let $\{f_n\}$ be a sequence in L^p such that $\sum_1^\infty \|F_n\| = m < \infty$.

Put $g_n(x) = \sum_1^n |f_r(x)|$.

At every point x the increasing sequence $g_n(x)$ has a limit (finite or infinite). Denote this limit by $g(x)$ then $g(x)$ is measurable and

$$\int |g(x)|^p \, dx = \lim_{n\to\infty} \int |g_n(x)|^p \, dx \leq m^p$$

therefore $g(x) \int L^P$ and $g(x) < \infty$ p.p. therefore $\sum_1^\infty f_n(x)$ is absolutely convergent p.p. to some function $f(x)$.

$$\left| f(x) - \sum_1^n f_r(x) \right|^p \leq \left(\sum_{n+1}^\infty |f_r(x)) \right)^p \leq (g(x))^p$$

Since $(g(x)) \in L^p$

$$\lim \int \left(\sum_{n+1}^\infty |f_r(x)| \right)^p \, dx = \int \lim \sum_{n+1}^\infty |f_r(x)|^p \, dx = 0$$

by Lebesgues's Dominated Convergence theorem therefore

$\int |f(x) - \sum_1^n f_r(x)|^p \to 0$ as $n \to \infty$ therefore

$||f(x) - \sum_1^n f_r(x)|| \to 0$ as $n \to \infty$.

Hence by the Lemma L^P is complete.

Other examples of Banach Spaces (i) The set $C(X)$ of all real-valued, or complex valued, functions defined and continuous on a complex space X where $\|F\| = \sup\{|f(x)| : x \in X\}$.

Convergence in this norm is uniform.

(ii) Set of all functions $f(z)$ analytic on the unit disc with

$$\|F\| = \sup\{|f(z)| : |z| \leq 1\}$$

(iii) Set of all functions $f(z)$ analytic on the unit disc with

$$\|F\| = \iint_{|z|\leq 1} |f| \, dxdy$$

(iv) The set of all functions $f(z)$ harmonic on the unit circle with

$$\|f\| = \sup\{|f(z)| : |z| \leq 1\}$$

[$f(z)$ is harmonic if $\frac{1}{\ell}\int_C f(z)\, dz = f(z_0)$ $\;C = \{z : |z - z_0| = \frac{\ell}{2\pi}\}$].