FUNCTIONAL ANALYSIS
INTEGRATION-FUNCTIONAL APPROACH

The starting-point is a vector lattice L of functions defined on a space X.

We have a functional I defined on L with the following properties

(i) $I : L \to R$

(ii) I is linear on L

(iii) $f \geq 0 \Rightarrow I(f) \geq 0$

(iv) If $f_n \downarrow 0$ $I(f_n) \to 0$

we now extend I to a wider class of functions $U = \{g : X \to \overline{R} : f_n \uparrow g$ for some sequence.$\}$

Suppose $f_n \uparrow g$ and $g_n \uparrow g$ where $\{f_b\}\{g_n\} \in L$.

For every n_0

$f_n \geq f_n \wedge g_{n_0} \uparrow g_{n_0}$ as $n \to \infty$ therefore

$$I(g_{n_0}) = \lim I(f_n \wedge g_{n_0}) \leq I(f_n)$$

Hence $\lim I(g_n) \leq \lim I(f_n)$.

Similarly $\lim I(g_n) \geq \lim I(f_n)$. Therefore

$$\lim I(g_n) = \lim I(f_n) =^{df} I_h$$

U is no longer a vector lattice, but it has the following properties:

(i) $f, g \in U \Rightarrow f + g \in U$

(ii) $f \in U$ $c \geq 0 \Rightarrow cf \in U$

(iii) $f, g \in U \Rightarrow f \vee g$ $f \wedge g \in U$.

We have the following results.

If $g_n \geq 0$ and $g_n \subset U$ and if $\sum g_n = g$ then $g \subset U$ and $I(g) = \sum I g_n$, for $g \geq 0$ $g \subset U \Leftrightarrow \exists \{f_n\} \subset L$ such that $f_n \geq 0$ and $g = \sum_1^\infty f_n$. In this case $I(g) = \sum I f_n$.

If $g_n \geq 0$ and $g_n \subset U$ and if $g = \lim g_n$ then $g \subset U$ and $I(g) = \lim I(g_n)$.

We now extend the definition of I to a different class of functions.

Define

$$
\begin{aligned}
\overline{I}(h) &= \inf\{I(g)\ g \in U\ g \geq h\} \\
\underline{I}(h) &= -\overline{I}(-h)
\end{aligned}
$$

(i) $\overline{I}(f+g) \leq \overline{I}(f) + \overline{I}(g)$

(ii) $\overline{I}(cf) = c\overline{I}(f)\ c \geq 0$

(iii) $f \leq g \Rightarrow \overline{I}(f) \leq \overline{I}(g)$ and $\underline{I}(f) \leq \underline{I}(g)$

(iv) $\underline{I}(f) \leq \overline{I}(f)$,

for $0 = \overline{I}(0) = \overline{I}(f - f) \leq \overline{I}(f) + \overline{I}(-f)$ therefore $I(f) = -\overline{I}(-f) \leq \overline{I}(f)$.

Theorem If $f \in U$ $\underline{I}(f) = \overline{I}(f) = I(f)$.

Proof If $f \in U$ clearly $\overline{I}(f) = I(f)$.

$\exists\{f_n\} \subset L$ such that $f_n \uparrow f$ therfore $-f_n \downarrow -f$

$$
\begin{aligned}
I(f) &= \lim I(f_n) \\
-I(f) &= \lim I(-f_n) \geq \overline{I}(-f) = -\underline{I}(f) \\
\text{therefore } \underline{I}(f) &\geq I(f)
\end{aligned}
$$

Hence the result.

Theorem If f_n is a sequence of non-negative functions and $f = \sum f_n$

$$\overline{I}(f) \leq \sum \overline{I}(f_n).$$

Proof Suppose without loss of generality $\sum \overline{I}(f_n) < \infty$.

Given $\varepsilon > 0$, for each n we can find $g_n \in U$ such that $g_n \geq f_n$ and $I(g_n) < \overline{I}(f_n) + \frac{\varepsilon}{2^n}$.

If $g = \sum g_n,\ g \in U\ g \geq f$ so

$$
\begin{aligned}
I(g) &\leq \sum \overline{I}(f_n) + \varepsilon \\
\text{therefore } \overline{I}(f) &\leq \sum \overline{I}(f_n) + \varepsilon \\
\text{therefore } \overline{I}(f) &\leq \sum \overline{I}(f_n)
\end{aligned}
$$

Define $L' = \{f : \overline{I}(f) = \underline{I}(f) < \infty\}$.

L' contains all functions of U on which I is finite. $L' \subset L$.

If $f \in L'$ define $I(f) = \overline{I}(f) = \underline{I}(f)$.

L' is a vector lattice.

Let $\circ = + \wedge$ or $\vee$.

Let $f, g \in L',\ \varepsilon > 0$

$\exists f_1\ f_2 : f_1 \in U\ f_2 \in -U\ f_2 \leq f \leq f_1$ and $I(f_1) + I(-f_2) < \varepsilon$.

$\exists g_1\ g_2 : g_1 \in U\ g_2 \in -U\ g_2 \leq g \leq g_1$, and $I(g_1) + I(-g_2) < \varepsilon$

$$f_2 \circ g \leq f \circ g \leq f_1 \circ g\ \ f_1 \circ g_1 \in U\ \ f_2 \circ g_2 \in -U$$
$$I(f_1 \circ g_1) + I(-f_2 \circ g_2) \leq I(f_1 - f_2) + I(g_1 - g_2) < 2\varepsilon$$

Therefore $f \circ g \in L'$

Scalar multiplication is trivial.

Theorem If $\{f_n\}$ is an increasing sequence of functions in L', if $\lim I(f_n) < \infty$ and if $= \lim f_n$ then $f \in L'$ and $I(f) = \lim(f_n)$.

Proof We may suppose that $f_1 = 0$ $f = \sum_1^\infty (f_{n+1} - f_n)$ therefore

$$\overline{I}(f) \leq \sum_1^\infty I(f_{n+1} - f_n) = \lim I(f_n)$$

Since $f \geq f_n$ for every n

$$\underline{I}(f) \geq \underline{I}(f_n) = I(f_n)$$

Therefore $\underline{I}(f) \geq \lim I(f_n)$ hence the result.

Theorem (Fatou's Lemma) Let f_n be a sequence of non-negative integrable ($\in L_1$) functions. Then $\inf f_n \in L'$. If $\underline{\lim} I(f_n) < \infty$ then $\underline{\lim} f_n \in L'$ and

$$I(\underline{\lim} f_n) \leq \underline{\lim} I(f_n)$$

Proof Define $g_n = f_1 \wedge f_2 \wedge \ldots \wedge f_n$

$g_n \in L'$ and $g_n \downarrow \inf f_n$ therefore $-g_n \uparrow -\inf f_n$ therefore $-\inf f_n \in L'$ therefore $\inf f_n \in L'$.

Define $h + n = \inf_{r \geq n} f_r \quad h_n \in L'$

$h_n \uparrow \underline{\lim} f_n$ therefore provided $\underline{\lim} I(f_n) < \infty$ $\underline{\lim} f_n \in L'$

$$\begin{aligned} I(h_n) &\leq I(f_r) \; r \geq n \\ \text{therefore } I(h_n) &\leq \underline{\lim} I(f_n) \\ \text{therefore } I(\underline{\lim} h_n) &\leq \underline{\lim} I(f_n) \end{aligned}$$

Theorem (Dominated Convergence) If $\{f_n\}$ is a sequence of integrable functions such that $|f_n| \leq g$ for some $g \in L'$, for every n and if $f_n \to f$ as $n \to \infty$ then $f \in L'$ and $I(f_n) \to I(f)$ as $n \to \infty$.

Proof $o \leq g + f_n \leq 2g$ therefore applying Fatou's Lemma to this sequence

$$\begin{aligned} I(\underline{\lim}|g + f_n) &\leq \underline{\lim} I(g + f_n) \\ \text{therefore } I(g + f) &\leq I(g) + \underline{\lim} I(f_n) \; f \in L' \\ \text{therefore } I(f) &\leq \underline{\lim} I(f_n) \\ \text{therefore } I(-f) &\leq \underline{\lim} I(-f_n) \\ &= -\overline{\lim} I(f_n) \\ \text{therefore } I(f) &\geq \overline{\lim} I(f_n) \end{aligned}$$

Hence the result.

this approach ties up with the measure approach in the following sort of way.

$f \geq 0 \; f : X \to R$. f is said to be measurable if $f \wedge g \in L'$ for every $g \in L'$.

The measurable functions constitute a vector lattice in which $\lim f_n$ is measurable. A subset Y of x is called a measurable set if X_Y is a measurable function. The measurable sets constitute a σ-algebra of sets. If we assume that $f \wedge 1 \in L'$ then $\{x : f(x) > a\}$ is measurable. If we define $\mu(Y) = I(X_Y)$ then for $f \in L' \quad \int f \, d\mu = I(f)$.