Question

Determine whether or not there exists a number $\alpha > 0$, so that there exists a hyperbolic triangle T whose interior angles are $\frac{\pi}{3}$, $\frac{\pi}{5}$, and α, and whose hyperbolic area area(T) is $\frac{\pi}{25}$. If such an α exists, determine its value (or values).

Answer

By the Gauss-Bonnet Formula:
area$(\tau) = \pi -$ (sum of interior angles)
and so

$$\begin{aligned} \text{area}(\tau) &= \pi - \left(\frac{\pi}{3} + \frac{\pi}{5} + \alpha\right) \\ &= \pi\left(1 - \frac{1}{3} - \frac{1}{5}\right) - \alpha \\ &= \pi\frac{7}{15} - \alpha \end{aligned}$$

Since the only requirement is that area$(\tau) > 0$, there is such an α, namely $\alpha = \left(\frac{7}{15} - \frac{1}{25}\right)\pi = \underline{\frac{32}{75}\pi}$