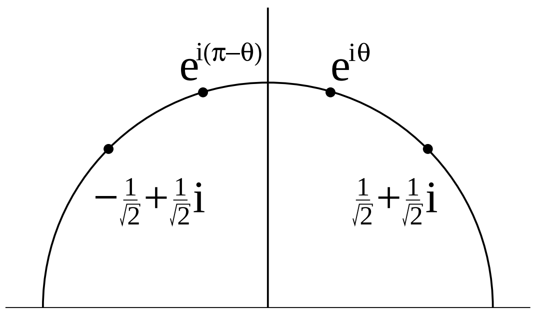
Question

Let ℓ be the closed hyperbolic line segment in the upper half-plane $\mathbf{H}$ joining $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$. Determine the two points which break ℓ into three segments of equal hyperbolic length. [Note that you won't be able to find these points explicitly, but you will be able to derive conditions determining them.]

Answer



By symmetry, the two points to be calculated are $e^{i\theta}$ and $e^{i(\pi-\theta)}$ where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ (since $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = e^{i\frac{\pi}{4}}$.

The hyperbolic length of the line segment joining $e^{i\frac{\pi}{4}}$ and $e^{i\theta}$ is

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\theta} \frac{1}{\sin(t)}\,dt &= \ln|\csc(t) - \cot(t)|\Big|_{\frac{\pi}{4}}^{\theta} \\ &= \ln\frac{|\csc(\theta) - \cot(\theta)|}{|\csc(\frac{\pi}{4}) - \cot(\frac{\pi}{4})|} \\ &= \ln\frac{|\csc(\theta) - \cot(\theta)|}{\sqrt{2} - 1}\end{aligned}$$

(Parametrizing the line segment by $z(t) = e^{it}$ and using that the element of arc length is $\frac{1}{\text{Im}(z)}|\,dz|$)

The hyperbolic length of the line segment joining $e^{i\theta}$ and $e^{i(\pi-\theta)}$ is twice the length of the line segment joining $e^{i\theta}$ and $e^{i\frac{\pi}{2}}$,
which is then

$$\begin{aligned} 2\int_{\theta}^{\frac{\pi}{2}} \frac{1}{\sin(t)}\,dt &= 2\ln|\csc(t)-\cot(\theta)|_{\theta}^{\frac{\pi}{2}} \\ &= 2\ln\frac{|\csc(\frac{\pi}{2})-\cot(\frac{\pi}{2})|}{|\csc(\theta)-\cot(\theta)|} \\ &= 2\ln\frac{1}{|\csc(\theta)-\cot(\theta)|} \end{aligned}$$

Hence, θ satisfies

$$\ln\frac{(\csc(\theta)-\cot(\theta))}{\sqrt{2}-1} = \ln\frac{1}{(\csc(\theta)-\cot(\theta))^2}$$

and so $(\csc(\theta)-\cot(\theta))^3 = \sqrt{2}-1$
$(1-\cos(\theta)^3 = (\sqrt{2}-1)\sin^3(\theta)$
(far enough since they don't have calculations.)