Question

Let **A** be the Euclidean circle given by the equation

$$3z\overline{z} + 10iz - 10i\overline{z} + 4 = 0$$

and let

$$m(z) = \frac{1}{z-1}.$$

Determine whether $m(\mathbf{A})$ is a Euclidean circle or the union of a Euclidean line with $\{\infty\}$. In the former case, determine its Euclidean centre and Euclidean radius. In the latter case, give its slope and the y-intercept.

Answer

Determine the equation for $m(A)$: [15 points]

set $w = m(z) = \dfrac{1}{z-1}$ and solve for z, so that $z = 1 + \dfrac{1}{w}$.

Substitute into the equation for a and simplify:

$$\begin{aligned}
0 &= 3z\bar{z} + 10iz - 10i\bar{z} + 4 \\
&= 3\left(\frac{1}{w} + 1\right)\left(1 + \frac{1}{\bar{w}}\right) + 10i\left(\frac{1}{w} + 1\right) - 10i\left(\frac{1}{\bar{w}} + 1\right) + 4 \\
&= 3\frac{1}{w}\frac{1}{\bar{w}} + (3 + 10i)\frac{1}{w} + (3 - 10i)\frac{1}{\bar{w}} + 7 \\
&= \frac{1}{w\bar{w}}\left(3 + (3 + 10i)\bar{w} + (3 - 10i)w + 7w\bar{w}\right)
\end{aligned}$$

So, the equation for $m(A)$ is

$$\underline{7w\bar{w} + (3 - 10i)w + (3 + 10i)\bar{w} + 3 = 0}$$

which is a euclidean circle (since the coefficient of $w\bar{w}$ is nonzero).

Determine the euclidean center and radius of $m(A)$. [10 points]

Complete the square:

$$\begin{aligned}
& w\bar{w} + \frac{3 - 10i}{7}w + \frac{3 + 10i}{7}\bar{w} + \frac{3}{7} \\
= & \left(w + \frac{3 + 10i}{7}\right)\left(\bar{w} + \frac{3 - 10i}{7}\right) + \frac{3}{7} - \frac{(3 + 10i)(3 - 10i)}{49} \\
= & \left|w - \left(\frac{-3 - 10i}{7}\right)\right|^2 - \frac{88}{49} = 0
\end{aligned}$$

So, the euclidean center is $\underline{\dfrac{-3 - 10i}{7}}$ and the euclidean radius is $\underline{\dfrac{\sqrt{88}}{7}}$.