Question

Decide which of the following series is convergent

(a) $1 + x + \frac{x^2}{2 \times 1} + \frac{x^3}{3 \times 2 \times 1} + ... + \frac{x^n}{n!} + ...$

(b) $100 + 10 + 1 + 0.1 + ... + 10^{2-n} + ...$

(c) $1 - 2 + 3 - 4 + ... + (-1)^{n-1}n + ...$

(d) $1 + 2 + 3 + 4 + ... + n + ...$

Answer

(a) $u_n = \frac{x^n}{n!}$ $\qquad S_n = 1 + x + ... + \frac{x^n}{n!}$

$\frac{u_{n+1}}{u_n} = \frac{x}{n+1} < r < 1$ for sufficiently large n.

Hence $\lim_{n\to\infty} S_n = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + ...$

$0 < S < 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}(1 + r + r^2 + ...)$ for some n

$\Rightarrow 0 < S < S_{n-1} + \frac{x^n}{n!(1-r)}$

So this is convergent (by d'Alembert's test)

(b) By the formula with $r = 0.1$ $u_1 = 100$

$S = \frac{100}{1 - 0.1} = \frac{u_1}{1-r} = \frac{100}{0.9}$

so this series is convergent

(c) $S = 1 - 2 + 3 - 4 + ...$ $\qquad S_n = \frac{n}{2}[1 + (-1)^n n]$

$|u_n|$ does not tend to zero, so S cannot exist Therefore this series is divergent.

(d) $S = 1 + 2 + 3 + 4 + ...$ $\qquad u_n = n$

$S_n = \frac{n}{2}[1 + n]$ $\qquad \lim_{n\to\infty} S_n = \infty$

So this series is divergent.