Question

Solve the following differential equations under the initial conditions given

(i) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 10x^2 + 3x + 17$ if $y = 4$, $\dfrac{dy}{dx} = 0$ when $x = 0$.

(ii) $\dfrac{d^2y}{dx^2} + 9y = 12\cos 3x$ if $y = 0$, $\dfrac{dy}{dx} = 9$ when $x = 0$.

Answer

(i) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 10x^2 + 3x + 17$

CF+PI type solution:

CF $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$

so auxiliary equation is $k^2 + 2k + 5 = 0 \Rightarrow k = -\dfrac{2 \pm \sqrt{4 - 4\cdot 5}}{2}$

so $k = -1 \pm 2i$

Thus we have (see notes)

CF: $\underline{y = e^{-x}(C\cos 2x + D\sin 2x)}$

For PI we have $f(x) = 10x^2 + 3x + 17$ so try PI

$$\begin{aligned} y &= L + Mx + Nx^2 \\ \frac{dy}{dx} &= M + 2Nx \\ \frac{d^2y}{dx^2} &= 2N \end{aligned}$$

Substituting into full equation:

$2N + 2(M + 2Nx) + 5(L + Mx + Nx^2) = 10x^2 + 3x + 17$

$$\Rightarrow \begin{array}{lllll} (1) & 2N + 2M + 5L & = & 17 & \text{(numbers)} \\ (2) & 4N + 5M & = & 3 & \text{(coeffs of x)} \\ (3) & 5N & = & 10 & \text{(coeffs of x}^2\text{)} \end{array}$$

Thus from (3), $N = 2$

Thus from (2), $M = \dfrac{3 - 4\times 2}{5} = -1$

Thus from (1), $L = \dfrac{17 - 2\times(-1) - (2\times 2)}{5} = 3$

Thus the PI is

$$\underline{y = 3 - x + 2x^2}$$

Hence CF+PI is

$$\underline{y = e^{-x}(C\cos 2x + D\sin 2x) + 2x^2 - x + 3} \quad A$$

We now have to solve for the boundary conditions, i.e., identify $C,\ D$.

$$\frac{dy}{dx} = e^{-x}(-2C\sin 2x + 2D\cos 2x - C\cos 2x - D\sin 2x) - 4x - 1 \quad B$$

When $x = 0,\ y = 4$, so from A

$$\begin{aligned} & 4 = e^0(C\cos 0 + D\sin 0) + 2\cdot 0 - 0 + 3 \\ \Rightarrow\ & 4 = C + 3 \\ \Rightarrow\ & \underline{C = 1} \end{aligned}$$

When $x = 0,\ \dfrac{dy}{dx} = 0$, so from B

$$0 = e^0(-2C\sin 0 + 2D\cos 0 - C\cos 0 - D\sin 0) + 5\times 0 - 1$$
$$\Rightarrow 0 = 2D - C - 1$$

Hence from $C = 1$,

$$o = 2D - 2 \Rightarrow \underline{D = 1}$$

Thus the total solution is

$$\underline{y = e^{-x}(\cos 2x + \sin 2x) + 2x^2 - x + 3}$$

(ii) $\dfrac{d^2y}{dx^2} + 9y = 12\cos 3x$ is a CF+PI type solution

CF: $\dfrac{d^2y}{dx^2} + 9y = 0$

so auxiliary equation is

$k^2 + 9 = 0 \Rightarrow k = \pm 3i$

Thus we have (see notes)

$$\underline{y = C\cos 3x + D\sin 3x}$$

PF: In this case $f(x) = 12\cos 3x$ and $\cos 3x$ occurs in the CF. Thus the usual substitution $y = A\cos 3x + B\sin 3x$ fails since terms in A and B cancel and so their values can't be found.

Thus try

$$\begin{aligned} y &= Lx\sin 3x + Mx\cos 3x \\ \frac{dy}{dx} &= 3Lx\cos 3x + L\sin 3x - 3Mx\sin 3x + M\cos 3x \\ \frac{d^2y}{dx^2} &= -(9Lx + 6M)\sin 3x + (6L - 9Mx)\cos 3x \end{aligned}$$

Substitute into full equation:

$-\sin 3x(9Lx + 6M) + \cos 3x(6L - 9Mx)$

$+9Lx\sin 3x + 9Mx\cos 3x = 12\cos 3x$

Compare coeffs of $\sin 3x$: $-6M = 0 \Rightarrow \underline{M = 0}$

Compare coeffs of $\cos 3x$: $6L = 12 \Rightarrow \underline{L = 2}$

Thus PI: $\underline{y = 2x\sin 3x}$

Thus the CF+PI general solution is:

$$\underline{y = C\cos 3x + D\sin 3x + 2x\sin 3x} \quad A$$

Hence $\dfrac{Dy}{dx} = -3C\sin 3x + 3D\cos 3x + 2(3x\cos 3x + \sin 3x) \quad B$

so if $y = 0$ when $x = 0$, A gives

$$0 = C + 0 + 0 \Rightarrow \underline{C = 0}$$

and if $\dfrac{dy}{dx} = 9$ when $x = 0$, B gives

$$9 = 0 + 3D + 2 \times 0 \Rightarrow \underline{D = 3}$$

Thus the final solution is

$$\underline{y = 3 \sin 3x + 2x \sin 3x = (3 + 2x) \sin 3x}$$