Question

Solve the equations

(i) $\dfrac{d^2y}{dx^2} + 4y = 8$

(ii) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 4e^{3x}$

(iii) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x \sin x$

Answer

(i) $\dfrac{d^2y}{dx^2} + 4y = 8$ This is a CF+PI solution.

CF: $\dfrac{d^2y}{dx^2} + 4y = 0$,

auxiliary equation $\Rightarrow k^2 + 4 = 0 \Rightarrow \underline{k = \pm 2i}$

Hence CF is $\underline{y = C\cos 2x + D\sin 2x}$

PI: $\dfrac{d^2y}{dx^2} + 4y = 8$

a constant, so from notes try $y = const = \alpha$, say.

So, substituting into the full equation,

$$0 + 4\alpha = 8 \Rightarrow \alpha = 2$$

Thus the PI is $\underline{y = 2}$

The solution is CF+PI

$$\underline{y = C\cos 2x + D\sin 2x + 2}$$

(ii) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 4e^{3x}$ This is a CF+PI solution.

CF: $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 0$,

auxiliary equation $k^2 - 4k + 3 = 0 \Rightarrow (k-1)(k-3) = 0 \Rightarrow k = 1, 3$

Hence CF is $\underline{y = Ae^x + Be^{3x}}$

Now PI: Note that $f(x) = 4e^{3x}$, but e^{3x} occurs in the CF, thus we can't try a PI solution $y = Le^{3x}$,as L will turn out to be zero.

Thus try the solution $y = Lxe^{3x}$

$$\begin{aligned} y &= Lxe^{3x} \\ \frac{dy}{dx} &= Le^{3x}(3x+1) \\ \frac{d^2y}{dx^2} &= Le^{3x}(6+9x) \end{aligned}$$

Thus substitute into the full equation:

$Le^{3x}(6+9x) - 4Le^{3x}(3x+1) + 3Lxe^{3x} = 4e^{3x}$

$\Rightarrow L(5 + 9x - 12x - 4 + 3x) = 4$

$\Rightarrow L = 2$

Thus the PI is $\underline{y = 2xe^{3x}}$

Hence the general solution is CF+PI

$$\underline{y = Ae^x + Be^{3x} + 2xe^{3x}}$$

(iii) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x \sin x$

This is a CF+PI solution

CF: $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$, auxiliary equation is

$k^2 + 2k + 1 = 0 \Rightarrow (k+1)^2 = 0 \Rightarrow \underline{k = -1}$

Hence CF is (from notes)

$$\underline{y = (A + Bx)e^{-x}}$$

The PI: Here $f(x) = e^x \sin x$, so try a solution

$$\begin{aligned} y &= e^x(L\sin x + M\cos x) \\ \frac{Dy}{dx} &= e^x([L+M]\cos x + [L-M]\sin x) \\ \frac{d^2y}{dx^2} &= 2e^x(L\cos x - M\sin x) \end{aligned}$$

So substituting into the full equation,

$2e^x(L\cos x - M\sin x) + 2e^x([L+M]\cos x + [L-M]\sin x)$
$+e^x(L\sin x + M\cos x) = e^x \sin x$

So compare coeffs of $e^x \cos x$

$2L + 2L + 2M + M = 0 \Rightarrow 4L + 3M = 0 \quad (1)$

Compare coeffs of $e^x \sin x$

$-2M - 2M + 2L + L = 1 \Rightarrow 3L - 4M = 1 \quad (2)$

From (1) and (2) must solve simultaneously for L and M.

Take $3 \times (1) - 4 \times (2)$

$$\begin{array}{rl} 12L + 9M = & 0 \\ \underline{12L - 16M =} & \underline{4} \\ 25M = & -4 \end{array}$$

$\Rightarrow \underline{M = -\frac{4}{25}}$

Hence in (1), $4L = -3M = \frac{12}{25} \Rightarrow \underline{L = \frac{3}{25}}$

Thus PI is

$$\underline{y = \frac{e^x}{25}(3\sin x - 4\cos x)}$$

Hence the general solution is: CF+PI

$$\underline{y = (A + Bx)e^{-x} + \frac{e^x}{25}(3\sin x - 4\cos x)}$$