Question

Solve the equations

(i) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 1 + 4x^2$

(ii) $2\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = \sin 2x$

(iii) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 3e^{2x}$

Answer

(i) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 1 + 4x^2$

First solve for the complementary function (CF).

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

Try solution $y = Ae^{kx}$ etc.

$\Rightarrow$ auxiliary equation

$k^2 - k - 2 = 0$

$\Rightarrow (k-2)(k+1) = 0$

$\Rightarrow k = 2$ or $k = -1$

Hence CF is $\underline{y = Ae^{2x} + Be^{-x}}$

Now work out the particular integral.

$f(x) = 1 + 4x^2$ is second degree polynomial so try a PI

$$\begin{aligned} y &= L + Mx + Nx^2 \\ \frac{dy}{dx} &= M + 2Nx \\ \frac{d^2y}{dx^2} &= 2N \end{aligned}$$

Substitute into full equation:

$2N - (M + 2Nx) - 2(L + Mx + Nx^2) = 1 + 4x^2$

or $(2N - M - 2L) + (-2N - 2M)x - 2Nx^2 = 1 + 4x^2$

compare numbers	$2N - M - 2L$	$=$	1	1
compare coeffs x^1	$-2N - 2M$	$=$	0	2
compare coeffs x^2	$-2N$	$=$	4	3

$(3) \Rightarrow N = -2$

Then

$(2) \Rightarrow M = -N \Rightarrow M = +2$

Thus

$(1) \Rightarrow L = \left(\dfrac{-1 + 2N - M}{2}\right) = \dfrac{-1 - 4 - 2}{2} = -\dfrac{7}{2}$

Thus the PI is

$$\underline{y = -\frac{7}{2} + 2x - 2x^2}$$

(PI)

The full general solution is the CF+PI

Thus

$$\underline{y = Ae^{2x} + Be^{-x} - \frac{7}{2} + 2x - 2x^2}$$

(ii) $2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = \sin 2x$

First solve for the complementary function (CF) $2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$

Try solution $y = Ae^{kx}$ etc.

$\Rightarrow$ auxiliary equation

$2k^2 + k - 1 = 0$

$\Rightarrow (2k - 1)(k + 1) = 0$

$\Rightarrow k = \dfrac{1}{2}$ or $k = -1$

Hence CF is $\underline{y = Ae^{\frac{x}{2}} + Be^{-x}}$

Now work out the particular integral:

$f(x) = \sin 2x$, so try a PI

$$\begin{aligned} y &= L\sin 2x + M\cos 2x \\ \frac{dy}{dx} &= 2L\cos 2x - 2M\sin 2x \\ \frac{d^2y}{dx^2} &= -4L\sin 2x - 4M\cos 2x \end{aligned}$$

Substitute into full equation:

$2(-4L\sin 2x - 4M\cos 2x) + 2L\cos 2x_2 M\sin 2x$

$-(L\sin 2x + M\cos 2x) = \sin 2x$

$\Rightarrow (-8L - 2M - L)\sin 2x + (-8M + 2L - M)\cos 2x = \sin 2x$

Compare coeffs of $\sin 2x$: $-8L - 2M - L = 1$

Compare coeffs of $\cos 2x$: $-8M + 2L - M = 0$

$$\Rightarrow \left.\begin{array}{r} -9L - 2M = 1 \\ +2L - 9M = 0 \end{array}\right\} \begin{array}{l} (1) \\ (2) \end{array}$$

Solve these equations by taking $2 \times (1) + 9 \times (2)$

$$\begin{array}{r} -18L - 4M = 2 \\ \underline{18L - 81M = 0} \\ -85M = 2 \end{array}$$

$\Rightarrow \underline{M = -\frac{2}{85}}$

Thus in (2) $2L - 9\left(-\frac{2}{85}\right) = 0 \Rightarrow \underline{L = -\frac{9}{85}}$

Thus the PI is

$$\underline{y = -\frac{9}{85}\sin 2x - \frac{2}{85}\cos 2x}$$

Thus the general solution is the CF + PI

$$\underline{y = Ae^{\frac{x}{2}} + Be^{-x} - \frac{9}{85}\sin 2x - \frac{2}{85}\cos 2x}$$

(iii) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 3e^{2x}$ First solve for the complementary function(CF)

$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

Try solution $y = Ae^{kx}$ etc.

$\Rightarrow$ auxiliary equation is

$k^2 - 8k + 16 = 0$

$\Rightarrow (k-4)^2 = 0$

$\Rightarrow k = 4, 4$ twice

Two equal roots, so the CF is

$\underline{y = (A + Bx)e^{4x}}$

Now work out the particular integral

$f(x) = 3e^{2x}$, so try a PI

$$\begin{aligned} y &= Le^{2x} \\ \frac{dy}{dx} &= 2Le^{2x} \\ \frac{d^2y}{dx^2} &= 4Le^{2x} \end{aligned}$$

Substitute into the full equation

$4Le^{2x} - 8(2Le^{2x}) + 16Le^{2x} = 3e^{2x}$

$\Rightarrow Le^{2x}(4 - 16 + 16) = 3e^{2x}$

$\Rightarrow L = \frac{3}{4}$

Thus $\underline{y = \frac{3}{4}e^{2x}}$ is the PI

The general solution is the CF+PI

$\underline{y = (A + Bx)e^{4x} + \frac{3}{4}e^{2x}}$