Question

A perfect fluid has an equation of state given by $p = \rho^\gamma$, where p is the pressure, ρ is the density and γ is a positive constant. Show that in one dimension its density and speed $v(x,t)$ satisfy:

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = -\rho\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\gamma\rho^{\gamma-2}\frac{\partial p}{\partial x}.$$

Answer

Using conservation of mass gives: $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(v\rho) = 0$,

which gives $\dfrac{\partial \rho}{\partial t} + v\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial v}{\partial x} = 0$

Using conservation of momentum gives: $\dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$.

Now $p = \rho^\gamma$, so $\dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial(\rho^\gamma)}{\partial x} = -\gamma\rho^{\gamma-2}\dfrac{\partial \rho}{\partial x}$.