Question
Show that $c_*(S) \leq C^*(S)$.

Answer
Let S be a bounded set. Suppose there are systems of disjoint rectangles contained within S. Otherwise $c_*(S) = 0$ and the result is trivial.

Lemma If a rectangle R is decomposed by a grid into rectangles $R_1, \cdots R_n$
$|R| = \sum |R_n|$
Proof Let the original rectangle be defined by $\{\mathbf{x} : a_i \leq x_i \leq b_i\}$.

Each axis x_i is subdivided by points $x_{i_1}, \cdots x_{i_{n_i}}$, so that
$a_i = x_{i_1} < x_{i_2} < \cdots < x_{i_{n_i}} = b$
Thus we have the system of rectangles defined by $\{\mathbf{x} : x_{ij} \leq x_i \leq x_{ij+1}\}$,
$i = 1, \cdots n, \quad j = 1, \cdots n_i - 1$
$|R| = \Pi_{i=1}^n (b_i - a_i)$
$= \Pi_{i=1}^n (x_{i_{n_i}} - x_{i_1})$
$= \Pi_{i=1}^n \sum_{j=1}^{n_i-1} (x_{ij+1} - x_{ij})$
$= \sum_{j_i=1}^{n_i-1} \Pi_{i=1}^n (x_{ij_i+1} - x_{ij_i})$
$= \sum |R_i|$
Now let $\{R_i\}$ be an arbitrary finite system of rectangles covering S and let $\{R'_i\}$ be a disjoint system within S. We use all the a_i, b_i, a'_i, b'_i of R_i and R'_i to form a grid . Some of the covering rectangles will contribute more than once to the grid. There will be more members of the grid making up the cover than the linear system. Therefore $\sum |R_i| \supseteq \sum |R'_i|$. Hence the result.