POINT SET TOPOLOGY

Definition 1 A topological structure on a set X is a family $\mathcal{O} \subset \mathcal{P}(X)$ called open sets and satisfying

(O_1) $\mathcal{O}$ is closed for arbitrary unions

(O_2) $\mathcal{O}$ is closed for finite intersections.

Definition 2 A set with a topological structure is a topological space $(X, \mathcal{O})$

$$\cup_{\emptyset} = \cup_{i \in \emptyset} E_i = \{x : x \in E_i \text{for some } i \in \emptyset\} = \emptyset$$

so $\emptyset$ is always open by (O_1)

$$\cap_{\emptyset} = \cap_{i \in \emptyset} E_i = \{x : x \in E_i \text{for all } i \in \emptyset\} = X$$

so X is always open by (O_2).

Examples **(i)** $\mathcal{O} = \mathcal{P}(X)$ the discrete topology.

(ii) $\mathcal{O}\{\emptyset, X\}$ the indiscrete of trivial topology.
These coincide when X has one point.

(iii) $\mathcal{Q}$=the rational line.
$\mathcal{O}$=set of unions of open rational intervals

Definition 3 Topological spaces X and X' are homomorphic if there is an isomorphism of their topological structures i.e. if there is a bijection (1-1 onto map) of X and X' which generates a bijection of $\mathcal{O}$ and $\mathcal{O}$.

e.g. If X and X are discrete spaces a bijection is a homomorphism. (see also Kelley p102 H).

Definition 4 A base for a topological structure is a family $\mathcal{B} \subset \mathcal{O}$ such that every $o \in \mathcal{O}$ can be expressed as a union of sets of $\mathcal{B}$

Examples **(i)** for the discrete topological structure $\{x\}_{x \in X}$ is a base.

(ii) for the indiscrete topological structure $\{\emptyset, X\}$ is a base.

(iii) For $\mathcal{Q}$, topologised as before, the set of bounded open intervals is a base.

(iv) Let $X = \{0, 1, 2\}$

Let $\mathcal{B} = \{(0, 1), (1, 2), (0, 12)\}$. Is this a base for some topology on X? i.e. Do unions of members of X satisfy (O_2)?

$(0, 1) \cap (1, 2) = (1)$- which is not a union of members of $\mathcal{B}$, so $\mathcal{B}$ is not a base for any topology on X.

Theorem 1 A necessary and sufficient condition for $\mathcal{B}$ to be a base for a topology on $X = \cup_{o \in \mathcal{B}} o$ is that for each O' and $O'' \in \mathcal{B}$ and each $x \in O' \cap O'' \exists O \in \mathcal{B}$ such that $x \in O \subset O' \cap O''$.

Proof Necessary: If $\mathcal{B}$ is a base for $\mathcal{O}$, $O' \cap O'' \in \mathcal{O}$ and if $x \in O' \cap O''$, since $O' \cap O''$ is a union of sets of $\mathcal{B}$ $\exists O \in \mathcal{B}$ such that $x \in O \subset O' \cap O''$.

Sufficient: let $\mathcal{O}$ be the family of unions of sets of $\mathcal{B}$.

(O_1) is clearly satisfied.

(O_2) $(\cup A_i) \cap (\cup B_j) = \cup (A_i \cap B_j)$ so that it is sufficient to prove that the intersection of two sets of $\mathcal{B}$ is a union of sets of $\mathcal{B}$.

Let $x \in O' \cap O''$. Then $\exists Ox \in \mathcal{B}$ such that $x \in O\ x \subset O' \cap O''$ so that $O' \cap O'' = \cup_{x \in O' \cap O''} Ox$.

Theorem 2 If $\mathcal{S}$ is a non-empty family of sets the family $\mathcal{B}$ of their finite intersections is a base for a topology on $\cup_{\mathcal{S}}$

Proof Immediate verification of O_1 and O_2. The topology generated in this way is the smallest topology including all the sets of $\mathcal{S}$.

Definition 4 A family $\mathcal{S}$ is a sub base for a topology if the set of finite intersections is a base for the topology.

e.g. $\{a \infty\}_{a \in Q}$ and $\{(-\infty\ a)\}_{a \in Q}$ are sub bases for $\mathcal{Q}$

Definition 5 If a topology has a countable base it satisfies the second axiom of countability.

Definition 6 In a topological space a neighbourhood of a set A is a set which contains an open set containing A. A neighbourhood of a point X is a neighbourhood of $\{x\}$.

Theorem 3 A necessary and sufficient condition that a set be open is that it contains (is) a neighbourhood of each of its points.

Proof Necessary: Definition of a neighbourhood

Sufficient: Let $O_A = \cup$ open subsets of A. O_A is open (O_1) and $O_A \subset A$.

If $x \in A$ $A \supset$ a neighbourhood of $x \supset$ open set $\ni x$ therefore $x \in O_A$ therefore $A \subset O_A$.

Let $V(x)$ denote the family of neighbourhoods of x. Then $V(x)$ has the following properties.

(V_1) Every subset of X which contains a member of $V(x)$ is a member of $V(x)$

(V_2) $V(x)$ is closed for finite intersections

(V_3) x belongs to every member of $V(x)$.

(V_4) If $v \in V(x) \exists W \in V(x)$ such that $v \in V(y)$ for all $y \in W$.
(Take W to be an open set $\ni x$ and $\subset V$.)

Theorem 4 If for each point x of X there is given a family $V(x)$ of subsets of X, satisfying V_{1-4} then $\exists$ a unique topology on X for which the sets of neighbourhoods of each point x are precisely the given $V(x)$. (Hausdorff).

Proof If such a topology exists theorem 3 shows that the open sets must be the O such that for each $x \in O$ $O \in V(x)$ and so there is at most one such topology. Consider the set O so defined.

(O_1) Suppose $x \in \cup O$. Then $x \in$some O' $O' \in V(x)$ $O' \subset |cupO$ so $\cup O \in V(x)$.

(O_2) Suppose $x \in \cap_F o \Rightarrow x \in$ each O. each $O \in V(x)$ therefore $\cap_F O \in V(x)$ by V_2.

Now consider the system $U(x)$ of neighbourhoods of x defined by this topology.

(i) $U(x) \subset V(x)$. Let $U \in U(x)$. Then $U \supset O \ni x$. But $O \in V(x)$ so by V_1 $U \in V(x)$.

(ii) $V(x) \subset U(x)$. Let $V \in V(x)$. It is sufficient to prove that $\exists O \in \mathcal{O}$ such that $x \in O \subset V$.
Let $O = \{y : V \in V(y)\}$ $x \in O$ since $V \in V(x)$.
$O \subset V$ since $y \in V$ for all $y \in O$ by V_3.
To prove that $O \in \mathcal{O}$ it is sufficient to prove that $V \in V(y)$ for all $y \in O$.
If $t \in O$ $V \in V(y)$ by definition of O therefore $\exists W \in V(y)$ such that $V \in V(z)$ for all $z \in W$ by $V - 4$.
Therefore $W \subset O$ (definition of O)

Therefore $O \in V(y)$ by V_1
e.g. $R_1 : V(x)=$ sets which contain interval (a,b) $a < x < b$.

Definition 7 A metric for a set X is a function ρ form $X \times X$ to R^+ (non-negative reals) such that

M_1 $\rho(x,y) = \rho(y,x)$ for all x, y

M_2 $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all x, y, z

M_3 $\rho(x,x) = 0$ for all x.

This is sometimes called a pseudo metric and for a metric we have

(M_3') $\rho(x,y) \geq 0, \; = 0 \Leftrightarrow x = y$.

Definition 8 The open r- Ball about x $= \{y : \rho(x,y) < r\}$ and is denoted by $B(r,x)$

The closed r-ball around x $= \{y : \rho(x,y) \leq R\}$ and is denoted by $\overline{B}(r,x)$.

$V(x) = \{$ sets which contain one of $B\left(\frac{1}{n}, x\right)$ $n = 1, 2, \ldots\}$ satisfies V_{1-4} and so defines a topology on X. This topology is the metric topology defined on X by ρ.

The development of topology from the neighbourhood point of ve=iew is due to H. Weyl and Hausdorff. That from the open sets aspect is due to Alexandroff and Hopf.

Definition 9 The closed sets G of a topological space are the complements of the open sets.

(G_1) G is closed for arbitrary intersections

(G_2) G is closed for finite unions. $\emptyset$ and X are both closed and open.

Clearly given a family G satisfying G_1 and G_2 the family of complements is a topology for which the closed sets are the sets of G.

Definition 10 a point x is an interior point of a set A if A is a neighbourhood of x.

The set of interior points of A is the interior A^0 of A.

An exterior point of A is an interior point of cA (i.e. $\exists$ a neighbourhood of x which does not meet A, or x is isolated, separated from A.)

Theorem 5 **(i)** The interior of a set is open and

(ii) is the largest open subset.

(iii) A necessary and sufficient condition for a set to be open is that it coincides with its interior.

Proof (i) $x \in A^0 \Rightarrow \exists$ open O such that $x \in O \subset A$.
$y \in O \Rightarrow y \in A^0$ therefore $x \in O \subset A^0$ therefore A^0 is a neighbourhood of each of its points and so it is open.

(ii) Let $O \subset A \Rightarrow O \subset A^o$ therefore $\cup_{O \subset A^0} O \subset A^o$, but A^0 is such an O therefore $\cup_{O \subset A} O = A^0$.

(iii) Sufficient condition from (i).
Necessary condition from (ii).
$\overbrace{A \cap B}^{0} = A^0 \cap B^0$ but $\overbrace{A \cup B}^{0} \neq A^0 \cup B^0$
e.g. $X = R_1$ with metric topology, A=rationals, B=irrationals.
$A^0 = \emptyset$ $B^0 = \emptyset$ therefore $A^0 \cup B^0 = \emptyset$. But $A \cup B = R_1$ and so $\overbrace{A \cup B}^{0} = R_1$.

However $A^0 \cup B^0 \subset \overbrace{A \cup B}^{0}$ always.

Definition 11 A point x is adherent to a set A if every neighbourhood of x meets A.

The set of points adherent to a set A is called the adherence (closure) $\overline{A}$ of A.

An adherent point of A is an isolated point of A if there is a neighbourhood of A which contains no point of A other than x; otherwise it is a point of accumulation (limit point) of A.

Examples (i) $A = \mathcal{Q} \subset \mathcal{R}$ $\overline{A} = R$

(ii) In a discrete space no set has accumulation points, every point is isolated.

(iii) In an indiscrete space every non-empty set has X as its adherence.

Theorem 5 (i) The adherence of a set is closed and

(ii) is the smallest closed set containing the given set

(iii) A is closed $\Leftrightarrow A = \overline{A} \Leftrightarrow A \supset$ its accumulation points.

$$\begin{aligned} c\overline{A} &= \overbrace{cA}^{0} \\ cA^0 &= \overline{cA} \end{aligned}$$

Corollary $\overline{\overline{A}} = \overline{A}$

$\overline{A} \cup \overline{B} = \overline{A \cup B}$

Definition 12 The set of accumulation points of A is its derived set A'.

A perfect set is a closed set without isolated points.

Suppose we map $\mathcal{P}(X) \to \mathcal{P}(X)$ where $A \to \overline{A}$ then

(C_1) $\overline{\emptyset} = \emptyset$

(C_2) $A \subset \overline{A}$

(C_3) $\overline{\overline{A}} \subset \overline{A}$ (with $C_2 \Rightarrow \overline{\overline{A}} = \overline{A}$)

(C_4) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Theorem 6 If we are given an operation mapping $\mathcal{P}(X)$ into $\mathcal{P}(X)$ which has the properties C_{1-4} then the set of complements of the sets G such that $G = \overline{G}$ is a topology on X for which $\overline{AS}$ is the closure of A for all $A \subset X$ (Kunatowski).

Definition 13 The frontier or boundary of a set A is $\overline{A} \cap \overline{cA}$ and is the set of points adherent to A and to cA, or is the set interior to neither A nor cA, or is the set neither interior or exterior to A. It is a closed set.

A set is closed $\Leftrightarrow$ it contains its boundary.

A set is open $\Leftrightarrow$ it is disjoint from its boundary.

Definition 14 A set is dense in X if $\overline{A} = X$.

A is dense in itself if all its points are accumulation point, i.e. $A \subset A'$.

A set is nowhere dense if $c\overline{A}$ is dense i.e. $\overline{A}^0 = \emptyset$.

Induced topologies Given $(X\mathcal{O})$ and $Y \subset X$, can we use $\mathcal{O}$ to get a topology for Y.

Theorem 7 $\{Y \cap O\}_{O \in \mathcal{O}}$ is a topology O_Y on Y called the topology induced on Y by $(x\mathcal{O})$

Proof (O_1) $\cup Y \cap O = Y_n \cup O$

$(O_2) \cap_F Y_n O = Y_n \cap_F O$

If $Z \subset Y \subset X$ then $\mathcal{O}_Y$ and $\mathcal{O}$ induce the same topology on Z.

A (sub) base $\mathcal{B}$ for $\mathcal{O}$ induces a (sub) base $\mathcal{B}_Y$ for $\mathcal{O}_Y$.

"O open in (relative to)Y" means"O open in (YO_Y).

A necessary and sufficient condition that every set open in Y be open in X is that Y be open in X.

Theorem 8 **(i)** $G \subset Y$ is closed in $Y \Leftrightarrow G = Y \cap G'$ where G' is closed in X.

(ii) $V_Y(x) = \{Y \cap V\}_{v \in V(x)}$

(iii) If $z \subset Y \subset X$ then $\overline{Z}_{mY} = Y_n \overline{Z}_{mx}$

Proof **(i)** G is closed in Y

$\Leftrightarrow Y \cap cG$ is open in Y

$\leftrightarrow Y \cap cG = Y \cap O; O$ open in X

$\Leftrightarrow Y \cap G = Y \cap cO$ (take comp. in Y)

$\Leftrightarrow G = Y \cap cO$

take $G' = cO$.

(ii) $Y \supset U \in V_Y(x)$

$\Leftrightarrow U \supset O \ni x$, O open in Y

$\Leftrightarrow U \supset O' \cap Y \ni x$ O' open in X

$\Leftrightarrow U = V \cap Y$ where $V \supset O' \ni x$

i.e. $U = Y \cap V$ where $V \in V(x)$.

(iii) $\overline{Z}_{mY} = \bigcap$ closed sets in Y which $\supset Z$

$= \bigcap(Y_\cap$ closed sets $\supset Z)$

$= Y \cap \bigcap$ closed sets $\supset Z$

$= Y \cap \overline{Z}$

Continuous functions

Definition 15 A map F of a topological space X into a topological space Y is continuous at $x_0 \in X$ if given a neighbourhood V of $f(x_0)$ in Y $\exists$ a neighbourhood U of x_+0 in X such that $f(U) \subset V$.

f is continuous at x_0 if for every neighbourhood V of $f(x_0)$ $f^{-1}(V)$ is a neighbourhood of x_0.

Theorem 9 If $f : X \to Y$ is continuous at X and $x \in \overline{A}$ then $f(x) \in \overline{f(A)}$

Proof Let $V \in V(f(x0)$ then $f^{-1}(V) \in V(x)$ therefore $f^{-1}(V) \cap A \neq \emptyset$. Therefore $f(f^{-1}(V)) \cap f(A) \neq f(\emptyset) = \emptyset$.

$f \circ f^{-1}$ is the identity therefore $V \cap f(A) \neq \emptyset$

Theorem 10 if $f : X \to Y$ is continuous at x_0 and $g : Y \to Z$ is continuous at $f(x_0)$ Then $g \circ f$ is continuous at x_0.

Proof Let $W \in V(g \circ f(x_0))$ then $g^{-1}(W) \in V(f(x_0))$

$f^{-1} \circ g^{-1}(W) \in V(x_0)$ $(g \circ f)^{-1}(W) \in V(x_0)$

Definition 16 A map $f : X \to$ is continuous (on X) if it is continuous at each point of X.

Theorem 11 Let $f : X \to Y$. Then the following properties are equivalent.

(i) f continuous on X

(ii) $f(\overline{A}) \subset \overline{f(A)}$ for all $A \in \mathcal{P}(X)$

(iii) the inverse image of a closed set is closed

(iv) the inverse image of an open set is open.

Proof (i)$\Rightarrow$ (ii) by theorem 9.

(ii)$\Rightarrow$ (iii) Let G' be closed in Y and $f^{-1}(G') = G$.

$f(\overline{G}) \subset \overline{f(G)}$ by (ii) $\subset \overline{G'} = G'$.

Therefore $\overline{G} \subset f^{-1}(G') = G \subset \overline{G}$ therefore $\overline{G} = G$ therefore $f^{-1}(G')$ is closed.

(iii)$\Rightarrow$ (iv)

$cf^{-1}(A) = f^{-1}(cA)$ $A \subset Y$

(iv)$\Rightarrow$ (i) Let $x \in X v \in V(f(x))$.

$\exists x \in O \subset V$ O open in Y. $f^{-1}(O)$ is open in Y and contains x so is a neighbourhood of x in $X f(f^{-1}(O)) \subset V$

Note The image of an open set under a continuous map is not necessarily open.

e.g. $f : R \to R$ $x \mapsto \frac{1}{1+x^2}$

$f(R) = (0\ 1]$ not open.

Comparison of Topologies

Definition 17 If $\mathcal{O}_1$ and $\mathcal{O}_2$ are topologies on a set X, $\mathcal{O}_1$ is finer than $\mathcal{O}_2$($\mathcal{O}_2$ is coarser then $\mathcal{O}$) if $\mathcal{O}_1 \supset \mathcal{O}_2$ (strictly finer if not equal).

[Topologies on X are not necessarily complete]

e.e. R_1 : $\mathcal{O}_1$:usual topology $\mathcal{O}_2$: open sets are open sets of $\mathcal{O}_1$ which contains O and $\{o\}$.

These topologies are not comparable.

Theorem 12 If $\mathcal{O}_1$ and $\mathcal{O}_2$ are topologies on X, the following properties are equivalent:

(i) $\mathcal{O}_1 \supset \mathcal{O}_2$

(ii) $\mathcal{O}_2$ - closed sets are $\mathcal{O}_1$ closed

(iii) $V_1(x) \supset V_2(x)$ for all $x \in X$

(iv) The identity map $(X\ \mathcal{O}) \to (X\mathcal{O}_2)$ is continuous.

(v) $\overline{A}^{(1)} \subset \overline{A}^{(2)}$ for all $A \subset X$

We also have the following qualitative results:

the discrete topology on a set is the finest topology on the set, and the indiscrete is the coarsest.

The finer the topology, the more open sets, closed sets, neighbourhoods of a point, the smaller the adherence, the larger the interior of a set, the fewer the dense sets.

If we refine the topology of X we get more continuous functions. If we refine the topology of Y we get fewer continuous functions.

Final Topologies

Theorem 13 Let X be a set. Let $(Y_i, \mathcal{O}_i = \{O_{ij}\}_{j \in J_i})_{i \in I}$ be a family of topological spaces.

Let $f_i : Y_i \to X$. Let $\mathcal{O} = \{O \subset X : f_i^{-1}(o) \text{ open in } Y_i \text{ for all } i \in I\}$.

Then $\mathcal{O}$ is a topology on X and is the finest for which the f_i are continuous.

If $g : X \to Z$ is a map into a topological space Z, g is continuous from $(X\mathcal{O}) \to Z \Leftrightarrow g \circ f_i$ are all continuous.

Proof $\mathcal{O}$ is non-empty: $f^{-1}(X) = Y_i$

(O_1) Let $f_i^{-1}(O_k) \in \mathcal{O}_1,\ O_k \in \mathcal{O}$. Then $f_i^{-1}(\cup_k O_k) = \cup_K f_i^{-1}(O_k) \in \mathcal{O}_1$ for all i.

(O_2) Let $f_i^{-1}(O - k) \in \mathcal{O}_i,\ O_k \in \mathcal{O}$. Then $f_i^{-1}(\cap_F O_k) = \cap_F f_i^{-1}(O_k) \in \mathcal{O}_i$ for all i.

A necessary and sufficient condition for f_i to be continuous is that $f_i^{-1}(O)$ be open in Y_i for all i. A finer topology than $\mathcal{O}$ will not satisfy this.

Now let $f_i : Y_i \to X\ \ g : X \to Z\ \ g \circ f_i : Y_i \to Z$.

The necessary condition is obvious.

Sufficient condition:

$g : X \to Y$ continuous $\Leftrightarrow g^{-1}(O)$ open in X for all O open in Z.

$g \circ f_i$ continuous $\Rightarrow g \circ {f_i}^{-1}(O)$ open in Y_i for all i

$\Rightarrow f_i^{-1} \circ g^{-1}(o)$ open in Y_i for all i

$\Rightarrow g^{-1}(O)$ is open in X

We define $\mathcal{O}$ to be the final topology for X, the maps f_i and spaces Y_i.

Examples **(i)** X a topological space. R is an equivalence relation on X.

$\phi : X \to \frac{X}{r} = Y \; x \mapsto \dot{x}$ (class)

The finest topology on Y such that ϕ is continuous is the quotient topology of that of X by the relation R.

$f : \frac{X}{R} \to Z$ is continuous $\Leftrightarrow f \circ \phi : X \to Z$ is continuous.

e.g. $X + R_2$

$\underline{R} : (x_1 \; y_1) \sim (x_2 \; y_2) \Leftrightarrow x_1 = x_2$.

Then $\frac{X}{\underline{R}} = R$ (isomorphically).

(ii) X a set. $(X \mathcal{O}_i)$ a family of topological spaces.

$\phi_i : (X, \mathcal{O}_i \to X \; x \mapsto x$

The final topology on X is the finest topology coarser than all the $\mathcal{O}_i$.

$\mathcal{O}$ is called the lower bound of the $\mathcal{O}_i$. $\mathcal{O} = \cap_i \mathcal{O}_i$

Initial Topologies

Theorem 14 Let X be a set. Let $(Y_i \mathcal{O}_i)_{i \in I}$ be a family of topological spaces. Let $f_i : X \to Y_i$. Let $f_i : X \to Y_i$. Let $S = \{f_i^{-1}(O_{ij})\}_{i \in I, j \in J_i}$.

Then S is a sub-base for a topology $\mathcal{J}$ on X, the coarsest topology for which all the f_i are continuous.

If Z is a topological space $g : Z \to X$ is continuous $\Leftrightarrow f_i \circ g$ continuous for all $i \in I$

Proof S is non-empty, for $\emptyset \in S$.

$\cup_{s \in S} S = X(-f_i^{-1}(Y_i))$ then use theorem 2.

A necessary and sufficient condition for f_i to be continuous is the $f_i^{-1}(O_{ij})$ be open in $\mathcal{J}$ for all ij.

the rest of the proof is similar to theorem 13.

We define $\mathcal{J}$ as the initial topology for X, the maps f_i and spaces Y_i.

Examples **(i)** X a set $(Y\mathcal{O})$ a topological space $f : X \to Y$.

the initial topology here is called the inverse image of $\mathcal{O}$.

(ii) If $X \subset Y$ $f : X \to Y$ $x \mapsto x$ is the canonical injection.

$f^{-1}(A) = A \cap X$ and the open sets of the initial topology are the intersections with X of the open sets of Y- we have the induced topology.

(iii) $(X\mathcal{O}_i)$ $\phi_1 : X \to X$ $x \mapsto x$.

The initial topology is the coarsest topology finer than all the $\mathcal{O}_i$

(iv) $(X_i\mathcal{O}_i)_{i\in I}$ $X = \prod_{i\in I} X_i$

$\phi_i = \text{proj}_i : X \to X_i$ $\{x_i\}_{i\in I} \mapsto x_i$

the initial topology is the coarsest for which all the projections are continuous and is called the product topology of the $\mathcal{O}_i$. $(X\mathcal{J})$ is the topological product of the $(X_i, \mathcal{O}_1)$. The $(X_i, \mathcal{O}_i)$ are the Factor spaces. The open sets of the product topology have as base the finite intersections of sets $\text{proj}_i^{-1}(O_{ij})$ where O_{ij} is open in $(X_i\ \mathcal{O}_i)$.

$\text{proj}_i^{-1}(O_{ij} = \prod_{i\in I} A_i$ where $A_u = X_u$ $i \neq j$, $A_j = O_{ij}$ and the base consists of sets $\prod_{i\in I} A_i$ where $A_i = X_i$ except for a finite set of i, where A_i is open in X_i. These are called elementary sets.

$g : Z \to \prod X_i$ is continuous $\Leftrightarrow$ $\text{proj}_i \circ g$ is continuous for all i i.e. all the co-ordinates are continuous.

Limit Processes Consider the following limit processes:

(i) $\lim_{n\to\infty} a_n$

(ii) $\lim_{x\to a} f(x)$

(iii) $R\int_a^b f(x)\,dx = \lim \sum (x_{i+1} - x_i) f(\xi_i)$

What have these in common.

A. A set with some order properties

(i) $Z \to R$ $n \mapsto a_n$

(ii) $V(a) \to R$ $v \mapsto f(v)$

(iii) Nets on $(a\ b)$ $N \to I(N, f)$

B. A map of the ordered set into a topological space.

C. We consider the set of images as we proceed along the order.

We now unify all these.

The Theory of Filters (H.CARTAN 1937)

Definition 18 A filter on a set X is a family $\mathcal{F} \subset \mathcal{P}(X)$ such that

(F_1) $A \in \mathcal{F}$ and $B \supset A \Rightarrow B \in \mathcal{F}$

(F_2) $\mathcal{F}$ is closed for finite intersections

(F_3) $\emptyset \notin \mathcal{F}$.

$F_3 \Rightarrow$ every finite A^n in F is non-empty.

$F_2 \Rightarrow X \in \mathcal{F}$ i.e. $\mathcal{F}$ is non-empty.

Examples (i) $X \neq \emptyset$ $\{x\}$ is a filter on X.

(ii) $\emptyset \neq A \subset X : \{Y : Y \supset A\}$ is a filter on X.

(iii) X an infinite set.

The complements of the finite subsets form a filter on X.

(iv) If $X = Z$ (the positive integers) the filter of (iii) is called the Fréchet filter.

(v) X a topological space. The set of neighbourhoods of $\emptyset \neq A \subset X$ is a filter. $A = \{x\}$ gives the neighbourhoods of x.

Comparison of Filters

Definition 19 If $\mathcal{F}$ and $\mathcal{F}'$ are filters on a set X and $\mathcal{F} \subset \mathcal{F}'$ we say $\mathcal{F}$ is coarser than $\mathcal{F}'$, $\mathcal{F}'$ is finer than $\mathcal{F}$.

Filters are not neccessarily comparable e.g. the filters of neighbourhoods of distinct points in a metric space.

If $\{\mathcal{F}_n\}_{i \in I}$ is a family of filters on X then $\mathcal{F} = \bigcap_{i \in I} \mathcal{F}_i$ is a filter.

Definition 20 The intersection of the $\mathcal{F}_i$ is the finest filter coarser than all the $\mathcal{F}_i$ and is the lower bound of the $\mathcal{F}_i$.

Theorem 15 Let $\mathcal{S}$ be a system of sets in X . A necessary and sufficient condition that $\exists$ a filter on X containing $\mathcal{S}$ is that the finite intersections of members of $\mathcal{S}$ be non-empty.

Proof Necessary: Immediate from F_2

Sufficient: Consider the family $\mathcal{F}$ of sets which contain a member of $\mathcal{S}'$, the set of finite intersections on $\mathcal{S}$.

$\mathcal{F}$ satisfies F_1, F_2, F_3.

Any filter $\supset \mathcal{S}$ is finer than $\mathcal{F}$. $\mathcal{F}$ is the coarsest filter $\supset \mathcal{S}$.

$\mathcal{S}$ is called a system of generators of $\mathcal{F}$.

Corollary 1 $\mathcal{F}$ is a filter on X, $A \subset X$. A necessary and sufficient condition that $\exists \mathcal{F}' \supset \mathcal{F}$ such that $A \in \mathcal{F}'$ is that $F \cap A \neq \emptyset \forall F \in \mathcal{F}$.

Corollary 2 A set Φ of filter on (non-empty) X has an upper bound in the set of all filters on $X \Leftrightarrow$ for every finite sequence.

$\{\mathcal{F}_i\}_{i=1,2,\ldots,n} \subset \Phi$ and every $A_i \in \mathcal{F}_i$ $i = 1, 2, \ldots n \cap A_i \neq \emptyset$.

Filter Bases If $\mathcal{S}$ is a system of generators for $\mathcal{F}$, $\mathcal{F}$ is not, in general, the set of subsets of X which contain an element of $\mathcal{S}$

Theorem 16 Given $\mathcal{B} \subset \mathcal{P}(X)$, a necessary condition that the family of subsets of X which contain an element of $\mathcal{B}$ be a filter is that $\mathcal{B}$ have the properties

(B_1) The intersection of 2 sets of $\mathcal{B}$ contains a set of $\mathcal{B}$.

(B_2) $\mathcal{B}$ is not empty; $\emptyset \notin \mathcal{B}$.

Definition 21 A system $\mathcal{B} \subset \mathcal{P}(X)$ satisfying B_1, B_2 is called a base for the filter it generates by Theorem 16.

2 filter bases are equivalent if they generate the same filter.

Theorem 17 $\mathcal{B} \subset \mathcal{F}$ is a base for $\mathcal{F} \Leftrightarrow$ each set of $\mathcal{F}$ contains a set of $\mathcal{B}$.

Theorem 18 A necessary and sufficient condition that $\mathcal{F}'$ with base $\mathcal{B}'$ be finer than $\mathcal{F}$ with base $\mathcal{B}$ id $\mathcal{B}' \subset \mathcal{B}$.

Examples **(i)** Let X be a non-empty partially ordered set $(\leq)$ in which each pair of elements has an upper bound. The sections $\{x : x \geq a\}$ of X form a filter base. The filter it defines is called the filter of sections of X.

(ii) X=set of nets on $[a, b]$ $N = (a = x_0 < x_1 < \ldots < x_n = b)$ $N_1 \leq N_2$ if $N_1 \subset N_2$.

An upper bound for N_1 and N_2 is $N_1 \cup N_2$.

(iii) The filter of neighbourhoods of O in R_1 has as bases $\left\{\left(-\frac{1}{n}, \frac{1}{n}\right)\right\}$, $\left\{\left[-\frac{1}{n}, \frac{1}{n}\right)\right\}$ $\{(-a, b)\ ab > 0\}$ etc.

(iv) In R_2 the squares, discs, ellipses et. centre 0 form bases for the filter of neighbourhoods of 0.

(v) In R_1 $\{x, |x| > n\}$ is a filter base.

Definition 22 A fundamental system of neighbourhoods of a point in a topological space is a base for the filter of neighbourhoods of the point.

Definition 23 A space satisfies the first axiom of countability if every point has a countable fundamental system of neighbourhoods.

2nd axiom $\Rightarrow$ 1st axiom

1st axiom $\not\Rightarrow$ 2nd axiom.

e.g. X uncountable, with discrete topology. $\{x\}$ is a base for the neighbourhoods of x and is countable.

Theorem 19 Let $\mathcal{F}$ be a filter on X $A \subset X$. A necessary and sufficient condition that $\mathcal{F}_A = \{F \cap A\}_{F \in \mathcal{F}}$ be a filter on A is that $F \cap A \neq \emptyset \forall F \in \mathcal{F}$.

If $A \in \mathcal{F}$ $\mathcal{F}_A$ is a filter on A.

Definition 24 $\mathcal{F}_A$ is a filter induced on A by $\mathcal{F}$

e.g. X a topological space $A \subset X$ $V_A(x)$ is a filter on $A \Leftrightarrow x \subset \overline{A}$

Let $f : X \to Y$ and let $\mathcal{F}$ be a filter on X. Then in general $\{f(F)\}_{F \in \mathcal{F}}$ is not a filter, for F_2 breaks down. But if $\mathcal{B}$ is a filter base for a filter on X then $\{f(B)\}_{B \in \mathcal{B}}$ is a base for a filter i=on Y.

Definition 25 X a topological space. $\mathcal{F}$ a filter on X. x is a limit point of $\mathcal{F}$ if $\mathcal{F} \supset V(x)$ we say $\mathcal{F}$ converges to X.

x is a limit of a filter base $\mathcal{B}$ if the filter with base $\mathcal{B}$ converges to x

e.g.

(i) In R_1, the filter with base $\left\{\left(-\frac{1}{n}, \frac{1}{n}\right)\right\}$ converges to 0, but that with base $\{\{x : |x| > n\}\}$ does not converge.

(ii)

$$\begin{aligned} X &= \{x, y, z\} \\ \mathcal{O} &= \{\emptyset, \{x, y\}, \{z\} X\} \\ \mathcal{B} &= \{\{x, y\} X\} \\ V(x) &= \{\{x, y\} X\} \end{aligned}$$

Therefore $\mathcal{B}$ converges to x, $V(y) = \{(x, y)\ X\}$ therefore $\mathcal{B}$ converges to y.

Definition 26 X a topological space.

$\mathcal{B}$ is a filter base on X. x is adherent to $\mathcal{B}$ if it is adherent to every set of $\mathcal{B}$.

x is adherent to $\mathcal{B} \Leftrightarrow V \cap B \neq \emptyset \forall V \in V(x),\ B \in \mathcal{B}$.

Every limit point of a filter is adherent to the filter.

The set of point adherent to a filter is $\cap_{B\in\mathcal{B}}\overline{B}$ and is closed.

Definition 27 Let X be a set, Y a topological space. Let $\mathcal{P} : X \to Y$. Let $\mathcal{F}$ be a filter on X.

$y \in Y$ is a limit of $\mathcal{F}$ along $\mathcal{F}$ if y is adherent to $\mathcal{P}(\mathcal{F})$

Examples (i) Fréchet $\mathcal{F}$ (of sections of Z) $\alpha : Z \to R \ \ n \mapsto a_n$.

A set of the filter is a set $\supset \{m : m \geq n\} = F \ \ \alpha(F_{\ni}\{a_m : m \geq n\}$.
a is a limit of α along $\mathcal{F} \Leftrightarrow a$ is adherent to every set of $\mathcal{F}$ i.e. every $(a-\varepsilon, a+\varepsilon)$ meets every set of $\mathcal{F}$ i.e. $(a-\varepsilon, a+\varepsilon) \cap \{a_m : m \geq n\} \neq \emptyset$ for all n.

(ii) X a topological space, Y a topological space. $\surd : X \to Y$. $V(a)$ is the filter of neighbourhoods of $a \in X$.
In this case we write $y = \lim\limits_{x\to a} \mathcal{P}(x)$ instead of $\lim\limits_{\mathcal{F}} \mathcal{P}$.

Theorem 20 X, Y topological spaces.

$\mathcal{P} : X \to Y$ continuous at $a \in Y \Leftrightarrow \lim\limits_{x\to a} \mathcal{P}(x) = \mathcal{P}(a)$.

Proof $\lim\limits_{x\to a} \mathcal{P}(x) = \mathcal{P}(a) \Leftrightarrow$ given $u \in U(\mathcal{P}(a)) \exists V \in V(a)$ such that $\mathcal{P}(V) \subset U$.

let X, Y be topological spaces. Let $A \subset X$ and let $a \in \overline{A}$. Let $f : A \to Y$. Let $\mathcal{F} = V_A(a) = \{A \cap V\}_{v \in V(a)}$.

We write $\lim\limits_{x\to a, x\in A} f(x)$ instead of $\lim\limits_{\mathcal{F}} f$.

Definition 28 $\lim\limits_{x\to a,\ x\in A} f(x)$ is a limit of f at a relative to A.

Theorem 21 Let X be a set and let $\{Y_i\}_{i\in I}$ be a family of topological spaces. Let $f_i : X \to Y_i$.

A filter $\mathcal{F}$ on X converges to $a \in X$ in the initial topology $\mathcal{O}$ on $X \Leftrightarrow$ the filter base $f_i(\mathcal{F})$ converges to $f_i(a)\ \forall i \in I$.

Proof Necessary condition: The f_i are continuous by Theorem 20.

Sufficient condition: Let $w \in V(a)$. By definition of $\mathcal{O} \exists \cap_J A_i \subset W$ where $a \in A_i = f^{-1}(O_i),\ O_i$ open in Y_i and J is a finite set.

Since $f_i(\mathcal{F})$ converges to $f_i(a)$ in Y_i $O_i \in f_i(\mathcal{F})$ and $f^{-1}(O_i) \in \mathcal{F}$ so that $\cap_J f^{-1}(O_i) \in \mathcal{F}$ i.e. $V(a) \subset F$.

Corollary A filter $\mathcal{F}$ on a product space $X = \prod_I X_i$ converges to $x \Leftrightarrow$ the filter base $pr_i(\mathcal{F})$ converges to x_i $\forall i \in I$.