Question

An incompressible viscous heat-conduction fluid on constant density ρ and constant kinematic viscosity ν flows past a flat plate at $y = 0$. The flow is two-dimensional. Far away from the plate, the speed of the fluid is $(U, 0)^T$ where U is a constant. YOU MAY ASSUME that for high Reynolds number steady boundary layer flow (with no body forces) in the region close to the plate the horizontal and vertical velocity components u and v satisfy the dimensional equations

$$\begin{aligned} uu_x + vu_y &= \underline{u}_{yy} \\ u_x + v_y &= 0 \end{aligned}$$

The fluid in the mainstream flow has temperature T_0, and the flat plate has temperature $T - 1 > T_0$. YOU MAY ALSO ASSUME that the temperature in the fluid obeys the energy equation

$$\rho c_p(uT_x + vT_y) = k(T_{xx} + T_{yy}) + \nu\rho\Phi$$

where

$$\Phi = 2u_x^2 + 2v_y^2 + v_x^2 + u_y^2 + 2v_x u_y$$

k and c_p are constants. By setting $T = T_0 + \overline{T}(T_1 - T_0)$, where $\overline{T}$ is a dimensionless temperature, and suitably scaling the other variables, show that the temperature in the boundary layer is determined by the dimensional equation

$$\rho c_p(uT_x + vT_y) = kT_{yy} + \rho\nu u_y^2.$$

(NOTE: you may assume that the quantities $k/(\mu c_p)$ and $U^2/(c_p(T_1 - T_0))$ are both $O(1)$.)

By seeking a similarity solution to the equations for u, v and T of the form

$$\begin{aligned} \psi &= \sqrt{\nu U x} f(\eta) \\ T &= T_0 + (T_1 - T_0) g(\eta) \end{aligned}$$

where ψ is the stream function so that $u = \psi_y$, $v = -\psi_x$ and the similarity variable η is given by

$$\eta = y\sqrt{\frac{U}{\nu x}},$$

show that f and g satisfy the ordinary differntial equations

$$\begin{aligned} f''' + \frac{1}{2} f f'' &= 0 \\ g'' + c_1 f g' + c_2 f''^2 &= 0 \end{aligned}$$

where $' = d/d\eta$ and c_1 and c_2 are constants that you should specify. Given suitable boundary conditions for f and g.

Answer

ASSUME $\begin{array}{lcl} uu_x + vu_y & = & \nu u_{yy} \\ u_x + v_y & = & 0 \end{array}$

And $\rho c_\rho(uT_x + vT_y) = k(T_{xx} + T_{yy}) + \nu\rho\frac{\partial q_i}{\partial x_j}\left(\frac{\partial q_j}{\partial x_i} + \frac{\partial q_i}{\partial x_j}\right)$

Now put $x = L\overline{x}$, $y = \epsilon L\overline{y}$, $u = U\overline{u}$, $v = \epsilon U\overline{v}$, $(p = \rho U^2\overline{p})$ and $T = T_0 + \overline{T}(T_1 - T_0)$

$\Rightarrow \frac{\rho c_\rho U(T_1 - T_0)}{L}(\overline{u}\overline{T}_{\overline{x}} + \overline{v}\overline{T}_{\overline{y}}) = \frac{k(T_1 - T_0)}{L^2}\left(\overline{T}_{\overline{x}\overline{x}} + \frac{1}{\epsilon^2}\overline{T}_{\overline{y}\overline{y}}\right) + \nu\rho\Phi$

$\Phi = 2u_x^2 + 2v_y^2 + v_x^2 + u_y^2 + 2v_x u_y$

$$\begin{aligned} & \frac{\rho c_\rho U(T_1 - T_0)}{L}\left(\overline{u}\overline{T}_{\overline{x}} + \overline{v}\overline{T}_{\overline{y}}\right) \\ = & \frac{k(T_1 - T_0)}{L^2}\left(\overline{T}_{\overline{x}\overline{x}} + \frac{1}{\epsilon^2}\overline{T}_{\overline{y}\overline{y}}\right) \\ & + \frac{\nu\rho U^2}{L^2}\left(2\overline{u}_{\overline{x}}^2 + 2\overline{v}_{\overline{y}}^2 + \epsilon^2\overline{v}_{\overline{x}}^2 + \frac{1}{\epsilon^2}\overline{u}_{\overline{y}}^2 + 2\overline{v}_{\overline{x}}\overline{u}_{\overline{y}}\right) \\ \overline{u}\overline{T}_{\overline{x}} + \overline{v}\overline{T}_{\overline{y}} = & \frac{k}{L\rho U c_\rho}\left(\overline{T}_{\overline{x}\overline{x}} + \frac{\overline{T}_{\overline{y}\overline{y}}}{\epsilon^2} + \frac{\nu U}{Lc_\rho(T_1 - T_0)}\left(2\overline{u}_{\overline{x}}^2 + 2\overline{v}_{\overline{y}}^2\right.\right. \\ & \left.\left. + \epsilon^2\overline{v}_{\overline{x}}^2 + \frac{1}{\epsilon^2}\overline{u}_{\overline{y}}^2 + 2\overline{v}_{\overline{x}}\overline{u}_{\overline{y}}\right)\right) \end{aligned}$$

Now we were told to assume that $|2|\mu c_\rho$, $U^2/c_p(T_1 - T_0)$ were $O(1)$. So TAKE THEM TO BE 1.

$$\begin{aligned} \Rightarrow \overline{u}\overline{T}_{\overline{x}} + \overline{v}\overline{T}_{\overline{y}} = & \left(\frac{\nu}{LU}\right)\left(\overline{T}_{\overline{x}\overline{x}} + \frac{1}{\epsilon^2}\overline{T}_{\overline{y}\overline{y}}\right) \\ & + \left(\frac{\nu}{LU}\right)\left(2\overline{u}_{\overline{x}}^2 + 2\overline{v}_{\overline{x}}^2 + \epsilon^2\overline{v}_{\overline{y}}^2 + \frac{1}{\epsilon^2}\overline{u}_{\overline{y}}^2 + 2\overline{v}_{\overline{x}}\overline{u}_{\overline{y}}\right) \end{aligned}$$

Now $\nu/LU = 1/Re$ and the assumption that was used to derive the momentum boundary layer equations in the fluid was $\epsilon^2 Re = 1$.
Thus for $\epsilon \ll 1$, $Re \gg 1$, $\epsilon^2 Re = 1$ we get
$\overline{u}\overline{T}_{\overline{x}} + \overline{v}\overline{T}_{\overline{y}} = \overline{T}_{\overline{y}\overline{y}} + \overline{u}_{\overline{y}}^2$. Redimensionalising $\Rightarrow$
$\rho c_p(uT_x + vT_y) = kT_{yy} + \rho\nu u_y^2$.
Now seek $\psi = \sqrt{vUx}f(\eta)$, $T = T_0 + (T_1 - T_0)g(\eta)$, $\quad\left(\eta = y\sqrt{\frac{U}{\nu x}}\right)$.
$u = \psi_y = Uf'(\eta)$, $u_y = \psi_{yy} = \frac{u^{3/2}}{\sqrt{\nu x}}f''$, $u_{yy} = \frac{U^2}{\nu x}f'''$

$$u_x = -\frac{1}{2}x^{-\frac{3}{2}}Uy\sqrt{\frac{U}{\nu}}f''$$

$$
\begin{aligned}
T_x &= (T_1 - T_0) y \sqrt{\frac{U}{\nu}} \left(-\frac{1}{2}\right) x^{-\frac{3}{2}} g' \\
T_y &= (T_1 - T_0) \sqrt{\frac{U}{\nu x}} g \\
T_{yy} &= (T_1 - T_0) \frac{U}{\nu x} g'' \\
v &= -\psi_x = -\frac{1}{2} x^{-\frac{1}{2}} \sqrt{U\nu} f - \sqrt{\nu U x} y (-\frac{1}{2} x^{-\frac{3}{2}}) y \sqrt{\frac{U}{\nu}} f'
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad & U f' \left(-\frac{1}{2} x^{-\frac{3}{2}} U^{\frac{3}{2}} \nu^{-\frac{1}{2}} f''\right) \\
& + U^{\frac{3}{2}} \nu^{-\frac{1}{2}} x^{-\frac{1}{2}} f'' \left(-\frac{1}{2} f x^{-\frac{1}{2}} U^{\frac{1}{2}} \nu^{\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}} y x^{\frac{1}{2}} U f'\right) \\
= \; & \frac{U^2}{x} f'''
\end{aligned}
$$

$$
\begin{aligned}
& \rho c_p \left(U f' (T_1 - T_0) y U^{\frac{1}{2}} \nu^{-\frac{1}{2}} \left(-\frac{1}{2}\right) x^{-\frac{3}{2}} g' \right. \\
& \left. + (T_1 - T_0) \sqrt{\frac{U}{\nu x}} g' \left(-\frac{1}{2} x^{-\frac{1}{2}} \sqrt{U} \nu f + \frac{1}{2} x^{-1} y U f'\right) \right) \\
= \; & (T_1 - T_0) \frac{U}{\nu x} g'' k + \rho \frac{\nu U^3}{v x} f''^2
\end{aligned}
$$

$\Rightarrow$

$$
U^2 x^{-1} \left(-\frac{1}{2}\right) f f'' = \frac{U^2}{x} f'''
$$

$$
\rho c_\rho (T_1 - T_0) \sqrt{\frac{U}{\nu x}} g' \left(-\frac{1}{2}\right) x^{-\frac{1}{2}} \sqrt{U\nu} f
$$

$$
= (T_1 - T_0) \left(\frac{U}{\nu x}\right) k g'' + \frac{\rho U^3}{x} f''^2
$$

i.e $\left.\begin{array}{l} f''' + \frac{1}{2} f f'' = 0 \\ g'' + c_1 f g' + c_2 f''^2 \end{array}\right\} \begin{array}{l} c_1 + \frac{1}{2} \frac{\mu c_\rho}{k} \\ c_2 = \mu U^2 / k (T_1 - T_0) \end{array}$

B/C's:- $f(0) = f'(0) =)$ (no slip), $f'(\infty) = 1$ (MATCHING)

$g(0) = 1,\ g(\infty) = 0$