Question
Define the Reynolds number Re for a viscous flow. YOU MAY ASSUME that a suitable non-dimensional form of the steady Navier-Stokes equations for use in studying boundary layer theory is

$$\begin{aligned}(\underline{q}.\nabla)\underline{q} &= -\nabla p + \frac{1}{Re}\nabla^2 \underline{q} \\ \nabla.\underline{q} &= 0\end{aligned}$$

where the fluid pressure and velocity are denoted respectively by p and $\underline{q} = (u, v)^T$, and lengths, pressures and velocities have been non-dimensionalised using L, ρU_∞^2 and U_∞ respectively. (Here ρ denotes the constant fluid density, and L and U_∞ are a typical length and speed in the flow.)
Starting from these equations, derive the non-dimensional boundary layer equations

$$\begin{aligned}uu_x + vu_y &= -p_x + u_{yy} \\ u_x + v_y &= 0\end{aligned}$$

for two-dimensional steady incompressible flow at high Reynolds number past a flat plate situated at $y = 0$.
Now assume that the horizontal speed of the external flow field is given (in dimensional variables) by

$$U(x) = U_\infty \left(\frac{x}{L}\right)^m,$$

where U_∞ and m are constants. By defining a stream function $\psi(x, y)$ which satisfies $u = \psi_y$, $v = -\psi_x$, show that ψ satisifies

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = mx^{2m-1} + \psi_{yyy}.$$

Give boundary conditions for ψ for this flow.
By assuming a solution of the form

$$\begin{aligned}\psi(x, y) &= x^{\frac{m+1}{2}} f(\eta) \\ \eta &= yx^{\frac{m-1}{2}}\end{aligned}$$

where $' = d/d\eta$, show that f satisfies the Falkner-Skan equation

$$f''' + \left(\frac{m+1}{2}\right) ff'' + m(1 - f'^2) = 0,$$

and give suitable boundary conditions for f.

Answer

ASSUME $\left.\begin{array}{ll}(\underline{q}.\nabla)\underline{q} & = -\nabla p + \dfrac{1}{Re}\nabla^2 \underline{q} \\ div(\underline{q}) & = 0\end{array}\right\}$

$Re = \dfrac{LU}{\nu}$ L : Typical length U : Typical speed
ν: Kinematic viscosicty

Now further rescale $y = \epsilon Y$, $v = \epsilon V$ (thin layer)

$$\Rightarrow \begin{array}{rcl} uu_x + Vu_y & = & -p_x + \dfrac{1}{Re}(u_{xx} + \dfrac{1}{\epsilon^2}u_{YY}) \\ \epsilon(uV_x + VV_Y) & = & \dfrac{-1}{\epsilon}p_Y + \dfrac{1}{Re}(\epsilon V_{xx} + \dfrac{1}{\epsilon}V_{YY}) \\ u_x + V_Y & = & 0 \end{array}$$

Now consider the relative size of ϵ and Re. If $Re\epsilon^2 \gg 1$ then for $Re \gg 1$ the first equation just gives Euler (no good). Also, if $Re\epsilon^2 \ll 1$ then it just gives $u_{YY} = 0 \Rightarrow u = A(x)Y$ which $\to \infty$ as $Y \to \infty \Rightarrow$ need $Re\epsilon^2 = O(1)$. Thus for $Re \gg 1$ get $p_Y = 0$ in the second equation $\Rightarrow p = p(x)$ alone, and $uu_x + Vu_Y = -p_x + u_{YY}$, $u_x + V_Y = 0$ or, back in N/D (unscaled) variables

$$\left.\begin{array}{rcl} uu_x + vu_y & = & -p_x + u_{yy} \\ u_x + v_y & = & 0 \end{array}\right\}$$

Now in the outer flow $U(x) = U_\infty \left(\dfrac{x}{L}\right)^m$. BUt here the flow is inviscid and so $p + \frac{1}{2}\rho U^2 = constant$ (Bernoulli)

$\Rightarrow p_x + \rho U U_x = 0$

i.e. $p_x = -\rho U_\infty^2 \left(\dfrac{x}{L}\right)^m m\dfrac{1}{L}\left(\dfrac{x}{L}\right)^{m-1}$

$\Rightarrow$ scaling p with ρU_∞^2, x with L gives (N/D)

$p_x = -mx^{2m-1}$

Now e have $uu_x + vu_y = mx^{2m-1} + u_{yy}$ and setting $u = \psi_y$, $v = -\psi_x$ we see that $u_x + v_y = 0$.

The momentum equation now gives

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = ms^{2m-1} + \psi_{yyy}$$

B/C's:- (no slip) $\psi = \psi_y$ at $y = 0$

$$\psi_y = x^m \text{ as y} \to \infty \text{ (MATCHING)}$$

So now try $\psi = x^{\frac{m+1}{2}}f(\eta)$, $\eta = yx^{\frac{m-1}{2}}$

$\psi_y = x^m f'$, $\psi_{yy} = x^{\frac{3m-1}{2}}f''$, $\psi_{yyy} = x^{\frac{4m-2}{2}}f'''$

$\psi_x = \left(\dfrac{m+1}{2}\right)x^{\left(\frac{m-1}{2}\right)}f + x^{\left(\frac{m+1}{2}\right)}y\left(\dfrac{m-1}{2}\right)x^{\frac{m-3}{2}}f'$

$\psi_{yx} = mx^{m-1}f' + x^m y\left(\dfrac{m-1}{2}\right)x^{\frac{m-3}{2}}f''$

$$
\begin{aligned}
\Rightarrow \qquad & x^m f'\left(m x^{m-1} f' + x^{\frac{3m-3}{2}} y \left(\frac{m-1}{2}\right) f''\right) \\
& - x^{\frac{3m-1}{2}} f'' \left(\left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}} f + x^{\frac{2m-2}{2}} y \left(\frac{m-1}{2}\right) f'\right) \\
= \quad & m x^{2m-1} + x^{2m-1} f'''
\end{aligned}
$$

$$\Rightarrow m x^{2m-1} f'^2 - \left(\frac{m+1}{2}\right) x^{2m-1} f f'' = m x^{2m-1} + x^{2m-1} f'''$$

$$\Rightarrow m f'^2 - \left(\frac{m+1}{2}\right) f f'' = m + f'''$$

i.e. $f''' + \left(\frac{m+1}{2}\right) f f'' + m(1 - f'^2) = 0$

B/C's:- $f(0) = f'(0) = 0$ (no slip)
$f'(\infty) = 1$ (MATCHING)