Question
A particle starts from rest and moves in a straight line with acceleration

$$\frac{dv}{dt} = a - kv^2$$

where v is the velocity and a and k are constants. Find the times taken to acquire a velocity V and show the distance travelled in this time is

$$x = \frac{1}{2k}\ln\left(\frac{a}{a - kV^2}\right)$$

(Hint: Note that if $v = \frac{dx}{dt}$, then$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$. This will enable you to change the differential equation above to one in terms of x and v only)

Answer
$\frac{dv}{dt} = a - kv^2$ variables separable
$\Rightarrow \int \frac{dv}{a - kv^2} = \int dt$
$\Rightarrow \int \frac{dv}{(\sqrt{a} + \sqrt{k}v)(\sqrt{a} - \sqrt{k}v)} = t + c$
Partial fractions
$\Rightarrow \frac{1}{2\sqrt{a}}\int \frac{dv}{(\sqrt{a} = \sqrt{k}v)} + \frac{1}{2\sqrt{a}}\int \frac{dv}{(\sqrt{a} = \sqrt{k}v)} = t + c$
$\Rightarrow \frac{1}{2\sqrt{ak}}\ln(\sqrt{a} + \sqrt{k}v) - \frac{1}{2\sqrt{ak}}\ln(\sqrt{a} + \sqrt{k}v) = t + c$
$\Rightarrow \ln\left(\frac{\sqrt{a} + \sqrt{k}v}{\sqrt{a} - \sqrt{k}v}\right) = 2\sqrt{ak}(t + c)$
$\Rightarrow \frac{\sqrt{a} + \sqrt{k}v}{\sqrt{a} - \sqrt{k}v} = e^{2\sqrt{ak}(t+c)}$
$\sqrt{a} + \sqrt{k}v = e^{2\sqrt{ak}(t+c)}(\sqrt{a} - \sqrt{k}v)$
$\sqrt{k}v(1 + e^{2\sqrt{ak}(t+c)}) = \sqrt{a}(e^{2\sqrt{ak}(t+c)} - 1) \Rightarrow v = \sqrt{\frac{a}{k}}\left(\frac{e^{2\sqrt{ak}(t+c)} - 1}{e^{2\sqrt{ak}(t+c)} + 1}\right)$
Now let $v = 0$ when $t = 0$ and $v = V$ when$t = T$
Thus

$$0 = \sqrt{\frac{a}{k}}\left(\frac{e^{2\sqrt{ak}c} - 1}{e^{2\sqrt{ak}c} + 1}\right)$$

So $e^{2\sqrt{ak}c} = 1$
$\Rightarrow 2\sqrt{ak}c = \ln 1 = 0$
If a and $k \neq 0$ we have $c = 0$.

Thus $v = \sqrt{\frac{a}{k}} \left(\frac{e^{2\sqrt{ak}t} - 1}{e^{2\sqrt{ak}t} + 1} \right)$
Thus $v = \sqrt{\frac{a}{k}} \left(\frac{e^{2\sqrt{ak}T} - 1}{e^{2\sqrt{ak}T} + 1} \right)$
or, after some algebra

$$\underline{T = \frac{1}{2\sqrt{ak}} \ln \left(\frac{\sqrt{a} + \sqrt{k}V}{\sqrt{a} - \sqrt{k}V} \right)}$$

Use hint and return to original equation
$\frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = a - kv^2$
$\Rightarrow v\frac{dv}{dx} = a - kv^2$
Also variables separable

$$\int \frac{vdv}{a - kv^2} = \int dx$$

Standard integral

$$\Rightarrow \frac{1}{-2k} \ln(a - kv^2) = x + c'$$

where c' is a new constant

$$\Rightarrow x = -c' + \frac{1}{2k} \ln \left(\frac{1}{a - kv^2} \right)$$

What are the boundary conditions?
Well $v = 0$ when $x = 0$
$\Rightarrow 0 = -c' = \frac{1}{2k} \ln \left(\frac{1}{a} \right)$
$\Rightarrow c' = \frac{1}{2k} \ln \left(\frac{1}{a} \right)$
$\Rightarrow x = -\frac{1}{2k} \ln \left(\frac{1}{a} \right) + \frac{1}{2k} \ln \left(\frac{1}{a - kv^2} \right)$
$\Rightarrow x = \frac{1}{2k} \ln \left(\frac{a}{a - kv^2} \right)$
Thus the distances travelled to when $v = V$ is

$$\underline{\frac{1}{2k} \ln \left(\frac{a}{a - kV^2} \right)} \text{ as required}$$