Question

An appeal fund is launched with a donation of 1,000; t weeks later the fund stands at A, and is growing at a rate of $Af(t)$, where

$$f(t) = \frac{1000t}{(t^2+125)^2}$$

models the growth and decline of enthusiasm of the sponsors. Write down the differential equation governing the appeal, find A in terms of t, and the time it takes to reach the target of $50,000$. To what value does the fund tend as $t \to \infty$?

Answer

Rate of growth of A: $\dfrac{dA}{dt} = \dfrac{A \cdot 1000t}{(t^2+125)^2}$

when $t = 0,\ A = 1000$

This is variables separable

$\displaystyle\int \frac{DA}{A} = \int \frac{1000t\,dt}{(t^2+125)^2}$

Use a substitution. Set $u = t^2 + 125,\ du = 2t + dt$

$\Rightarrow \ln A = \displaystyle\int 1000\frac{du}{2} \times \frac{1}{u^2}$

$\Rightarrow \ln A = 500\left[-\dfrac{1}{u}\right] + c$ where $u = t^2 + 125$

Thus $\ln A = c - \dfrac{500}{(t^2+125)}$

When $t = 0,\ A = 1000$

$\Rightarrow \ln(1000) = c - \dfrac{500}{125}$

$\Rightarrow c = \ln(1000) + 4$

Thus

$$\ln A = \ln(1000) + 4 - \frac{500}{(t^2+125)}$$

$\Rightarrow \ln\left(\dfrac{A}{1000}\right) = \dfrac{4t^2 + 500 - 500}{(t^2+125)}$

$\Rightarrow \ln\left(\dfrac{A}{1000}\right) = \dfrac{4t^2}{(t^2+125)}$

$\Rightarrow \underline{A = 1000\exp\left[\dfrac{4t^2}{t^2+125}\right]}$

When $A = 50,000$ we have

$$\frac{50,000}{1000} = \exp\left[\frac{4t^2}{t^2+125}\right]$$

$$\Rightarrow \ln(50) = \frac{4t^2}{t^2+125}$$

$$\Rightarrow t^2\ln(50) + 125\ln(50) = 4t^2$$

$$\Rightarrow t^2 = \frac{125\ln(50)}{4-\ln(50)}$$

or $\underline{t = \sqrt{\frac{125\ln(50)}{4-\ln(50)}} = 74.6 = 75 weeks}$

As $t \to \infty,\ A \to 1000\exp\left[\frac{4\times\infty}{\infty}\right] = 1000e^4 = 54,598 \approx 54,600$