Question

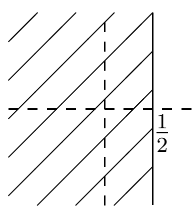
Show that if α has a positive imaginary part then the transformation

$$z \longrightarrow \frac{z-\alpha}{z-\bar{\alpha}}$$

maps the upper half plane onto the unit disc $D = \{z|\,|z| < 1\}$.
Hence find a transformation T which maps the half plane $\{z = x+iy|x \leq \frac{1}{2}\}$ onto D and maps 0 to 0 and ∞ to -1.
Find the image of the strip $\{z = x+iy|0 \leq x \leq \frac{1}{2}\}$ under T.

Answer

Let $w = \dfrac{z-\alpha}{z-\bar{\alpha}}$, if $z = x$ - real then $w = \dfrac{x-\alpha}{x-\bar{\alpha}} = \dfrac{x-\alpha}{\bar{x}-\bar{\alpha}}$, so $|w| = 1$.
Conversely if $|w| = 1$ then $|z-\alpha| = |z-\bar{\alpha}|$ i.e. z is equidistance from α and $\bar{\alpha}$ and is therefore real.
So the transformation maps the real axis to the unit circle.
Now $w = 0$ is the image of $z = \alpha$, so if im$\alpha > 0$, the interior of U maps to the interior of D.



$z \longrightarrow z - \frac{1}{2}$ $\quad$ $z \longrightarrow -iz$ The composite of these two maps sends H to$U$.

So $w = \dfrac{-i\left(z-\frac{1}{2}\right)-\alpha}{-i\left(z-\frac{1}{2}\right)-\bar{\alpha}}$ maps H to D.

$z = 0 \rightarrow w = 0 \Leftrightarrow \dfrac{1}{2}i - \alpha = 0$ i.e. $\alpha = \dfrac{1}{2}i$.

Then $w = \dfrac{-iz}{-iz+\frac{1}{2}i}$

Now under this transformation $z = \infty \rightarrow w = 1$
So a reflection will make $z = \infty \rightarrow w = -1$
i.e. $w \rightarrow -w$

So $w = \dfrac{iz}{-iz+\frac{1}{2}i}$

does all that is required.