Question

Let f have a pole of order 2 at $z = z_0$ and write $\phi(z) = (z - z_0)^2 f(z)$. Show that the residue of f at z_0 is equal to $\phi'(z_0)$.

Hence, or otherwise, evaluate by contour integration

i) $\displaystyle\int_0^\infty \frac{x^2 dx}{(1+x^2)^2}$

ii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$.

Answer

$f(z)$ has a pole of order 2 at z_0

So $f(z) = \dfrac{b_2}{(z-z_0)^2} + \dfrac{b_1}{z-z_0} + g(z)$

where $g(z)$ is analytic near z_0.

$(z-z_0)^2 f(z) = b_2 + b_1(z-z_0) + (z-z_0)^2 g(z)$

$\dfrac{d}{dz}(z-z_0)^2 f(z) = b_1 + 2(z-z_0)g(z) + (z-z_0)^2 g'(z) = b_1$ when $z = z_0$.

i) Let $f(z) = \dfrac{z^2}{(1+z^2)^2}$.

Let Γ be:

DIAGRAM

$f(z)$ has a pole of order 2 at $z = i$ inside Γ, with residue

$$\left.\frac{d}{dz}(z-i)^2 f(z)\right|_{z=i}$$

$$= \frac{d}{dz}\frac{z^2}{(z+i)^2} = \left.\frac{(z+i)^2 2z - z^2 2(z+i)}{(z+i)^4}\right|_{z=i} = -\frac{i}{4}$$

So $\displaystyle\int_\Gamma f(z)dz = -2\pi i \frac{i}{4} = \frac{\pi}{2}$

On the semi-circle $|f(z)| \le \dfrac{|z|^2}{(|z|^2-1)^2} = \dfrac{R^2}{(R^2-1)^2}$

So $\displaystyle\left|\int_C f(z)dz\right| \le \frac{R^2}{(R^2-1)^2} 2\pi R \to 0$ as $R \to \infty$

Thus $\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2}$ and so $\int_{0}^{\infty} \frac{x^2}{(1+x^2)^2}dx = \frac{\pi}{4}$.

ii) Let $z = e^{i\theta}$ and C be the unit circle

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$

$$I = \int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2} = \int_C \frac{dz}{iz\left(2+\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2} = \frac{4}{i}\int_C \frac{z}{(z^2+4z+1)^2}dz$$

Now $z^2 + 4z + 1 = 0$ if $z = z_0 = \frac{-4+2\sqrt{3}}{2}$ inside C

or $z = z_1 = \frac{-4+-\sqrt{3}}{2}$ outside C

So $f(z)$ has a pole of order 2 at $z = z_0$, with residue

$$\left.\frac{d}{dz}\frac{z}{(z-z_1)^2}\right|_{z=z_0} = \frac{(z_0-z_1)^2 - 2z_0(z_0-z_1)}{(z_0-z_1)^4}$$

$$= \frac{(2\sqrt{3})^2 - (-4+2\sqrt{3})2\sqrt{3}}{(2\sqrt{3})^4} = \frac{\sqrt{3}}{18}$$

So $I = \frac{4}{i}2\pi i\frac{\sqrt{3}}{18} = \frac{4\sqrt{3}}{9}\pi$