Question

State, without proof, the Cauchy-Riemann equations giving a necessary condition for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region A and state sufficient conditions, involving the Cauchy-Riemann equations, for f to be analytic in A.

Show that in polar co-ordinates the Cauchy-Riemann equations become

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \quad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.$$

Deduce that

$$(\cos(\log r) + i\sin(\log r))e^{-\theta}$$

defines an analytic function in some neighbourhood of each point other than the origin. Write this expression in the form $F(z)$, where $z = re^{i\theta}$, and discuss the multi-valued nature of F.

Choosing the branch of F for which $F(1) = 1$ compute $\int_\delta F(z)dz$, where δ is the upper half of the unit circle from $z = 1$ to $z = -1$.

Answer

The Cauchy-Riemann equations are $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ which must hold at each point of A. For sufficiency we need the additional conditions that the partial derivatives should be continuous in A.

Using the chain rule gives, with $x = r\cos\theta \;\; y = r\sin\theta$

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
\frac{\partial u}{\partial \theta} &= -\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta \\
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \\
\frac{\partial v}{\partial \theta} &= -\frac{\partial v}{\partial x}r\sin\theta + \frac{\partial v}{\partial y}r\cos\theta
\end{aligned}$$

So

$$\begin{aligned} r\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} r\cos\theta + \frac{\partial u}{\partial y} r\sin\theta \\ &= \frac{\partial v}{\partial y} r\cos\theta - \frac{\partial v}{\partial x} r\sin\theta = \frac{\partial v}{\partial \theta} \end{aligned}$$

Also

$$\begin{aligned} r\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x} r\cos\theta + \frac{\partial v}{\partial y} r\sin\theta \\ &= -\frac{\partial v}{\partial y} r\cos\theta + \frac{\partial u}{\partial x} r\sin\theta = -\frac{\partial u}{\partial \theta} \end{aligned}$$

$u = \cos(\log r)e^{-\theta} \qquad v = \sin(\log r)e^{-\theta}$

$$\begin{aligned} \frac{\partial u}{\partial r} &= -\sin(\log r)\frac{1}{r}e^{-\theta} \\ \frac{\partial v}{\partial \theta} &= -\sin(\log r)e^{-\theta} \text{ so } r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= \cos(\log r)\frac{1}{r}e^{-\theta} \\ \frac{\partial u}{\partial \theta} &= -\cos(\log r)e^{-\theta} \text{ so } r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta} \end{aligned}$$

The partial derivatives are continuous except at $z = 0$, so the u and v define an analytic function except at $z = 0$.
$(\cos(\log r) + i\sin(\log r))e^{-\theta} = e^{i\log r}e^{-\theta} = e^{i(\log r + i\theta)} = e^{i\log z} = z^i$
z^i is multi-valued because replacing $\log r$ by $\log r + 2n\pi$ gives the same answer.
On the unit circle $r = 1$, so with $F(z) = e^{-\theta}$ we have $F(1) = e^{-0} = 1$.
$\int_\delta F(z)dz = \int_0^\pi e^{-\theta} ie^{i\theta} d\theta$
$= \int_0^\pi ie^{i\theta(1+i)}d\theta = \left[\frac{e^{i\theta(1+i)}}{1+i}\right]_0^\pi = \frac{e^{i\pi(1+i)} - 1}{1+i} = \frac{-e^{-\pi} - 1}{1+i}$