Question

A surface $z = u(x, y)$ passes through a given closed curve C in x, y, z space whose projection on the x, y plane is Γ. Show that the area of that part of the surface enclosed by C is

$$A = \iint_S (1 + u_x^2 + u_y^2)^{\frac{1}{2}} dxdy$$

where S is the area in the plane bounded by Γ.
Show that when this area is a minimum, u satisfies the partial differential equation

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0.$$

(This is known as Plateau's problem).

Answer

For a surface $f(x, y, z) = 0$ a normal to the surface is given by $\underline{\nabla} f$, i.e., (f_x, f_y, f_z)
Hence for $u(x, y) - z = 0$ a normal vector is $(u_x, u_y, -1)$ and a unit normal is $(1 + u_x^2 + u_y^2)^{-\frac{1}{2}}(u_x, u_y, -1)$.
it follows that is ds is an elements of area of the surface, then by projection onto the $x - y$ plane,

$$(1 + u_x^2 + u_y^2)^{-\frac{1}{2}} ds = dxdy$$

and so

$$\int ds = \iint_S (1 + u_x^2 + u_y^2)^{\frac{1}{2}} dxdy$$

The E-L equation is then
$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$ with $F = (1 + u_x^2 + u_y^2)^{\frac{1}{2}}$.
This gives:

$$\frac{\partial}{\partial x}\left(\frac{u_x}{(1 + u_x^2 + u_y^2)^{\frac{1}{2}}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{(1 + u_x^2 + u_y^2)^{\frac{1}{2}}}\right) = 0$$

which after boring algebra gives the result in the question.