Question

(a) A curve C joins two points (a, α), (b, β) and has prescribed slopes at $x = a$, and b. Given that the functional

$$I = \int_a^b F(y, y', y'', x)dx$$

must be stationary when evaluated along this curve, write down the Euler-Lagrange equation which determines C.

(b) If F does not explicitly depend of x or y, show that the above equation for the extremal has a first integral

$$y'' \frac{\partial F}{\partial y''} - F = Ay' + B$$

where A, B are constants. (Hints: After a simplification, try a multiplication by y'' and then carry out a partial integration, as in the "special cases" section of the simple E-L notes. The note that $F = F(y', y'')$ only and recognise the first derivative of F wrt x).

(c) Derive a differential equation for the function $y(x0$ which makes

$$I = \int_0^2 y' y''^2 dx$$

stationary. Solve this equation, given the boundary conditions $y(0) = y'(0) = 0,\ y(2) = 1,\ y'(2) = 1$. (Hint: use the answer of part b above).

Answer

(a) Bookwork: $\dfrac{\partial F}{\partial y} - \dfrac{\partial}{\partial x}\left(\dfrac{\partial F}{\partial y'}\right) - \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial F}{\partial y''}\right) = 0$

(b) If $F = F(y', y'')$ then

$$-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) - \frac{\partial F}{\partial y'} = const = A \text{ say}$$

now multiply through by y'':

$$\Rightarrow \underbrace{\frac{d}{dx}\left(y''\frac{\partial F}{\partial y''}\right) - y'''\frac{\partial F}{\partial y''}}_{\text{spot this partial integration as in notes}} - y''\frac{\partial F}{\partial y'} = Ay''$$

$$\Rightarrow \frac{d}{dx}\left(y''\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}F(y', y'') = Ay''$$

$$\Rightarrow y''\frac{\partial F}{\partial y''} - F(y', y'') = Ay' + B \text{ as required.}$$

(c) $F = y'y''^2$ so (B) $\Rightarrow 2y'y''^2 - y'y''^2 = Ay' + B$ since $y'(0) = 0 \Rightarrow B = 0$

Therefore $y' = 0$ or $y'' = 2\alpha$ say $\Rightarrow y = \alpha x^2 + \beta x + \gamma$ $y(0) = 0,\ y'(0) = 0 \Rightarrow \gamma = \beta = 0;\ y(2) = 1 \Rightarrow \alpha = \dfrac{1}{4},\ y'(2) = 1$ is satisfied

$\Rightarrow$ solution is $\underline{y = \dfrac{x^2}{4}}$