Question

Find the general solution to the first order system of equations

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where A is the matrix given by

(a) $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

(a) $\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

Answer

(a) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues $\lambda = -1$ and $\lambda = 2$.

When $\lambda = -1$ the eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

When $\lambda = 2$ the eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Hence the solution is $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$.

(b) $A = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$ has eigenvalues $\lambda = 0$ and $\lambda = -2$.

When $\lambda = 0$ the eigenvector is $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

When $\lambda = -2$ the eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Hence the solution is $\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$.

(c) $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ has eigenvalues $\lambda = -1$ and $\lambda = 1$.

When $\lambda = -1$ the eigenvector is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

When $\lambda = 1$ the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Hence the solution is $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$.

(d) $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$ has eigenvalues $\lambda = -3$ and $\lambda = 2$.

When $\lambda = -3$ the eigenvector is $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

When $\lambda = 2$ the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Hence the solution is $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$.

(e) $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$ has eigenvalues $\lambda = \pm i$.

When $\lambda = i$ the eigenvector is $\begin{pmatrix} 2+i \\ 1 \end{pmatrix}$.

Hence the corresponding solution is given by

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} \\ &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos t + i \sin t) \\ &= \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \end{aligned}$$

Since the real and imaginary parts are each solutions to the equation we find the general solution is given by:
$\mathbf{x}(t) = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$.

(f) $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$ has eigenvalues $\lambda = -1 \pm 2i$

When $\lambda = -1 + 2i$ the eigenvector is $\begin{pmatrix} 2i \\ 1 \end{pmatrix}$.

Hence the corresponding solution is given by

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{-t+2it} \\ &= \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{-t}(\cos 2t + i\sin 2t) \\ &= \begin{pmatrix} -2e^{-t}\sin 2t \\ e^{-t}\cos 2t \end{pmatrix} + i \begin{pmatrix} 2e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{pmatrix} \end{aligned}$$

Since the real and imaginary parts are each solutions to the equation we find the general solution is given by:
$\mathbf{x}(t) = c_1 \begin{pmatrix} -2e^{-t}\sin 2t \\ e^{-t}\cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{pmatrix}$.

(g) $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$ has eigenvalues $\lambda = 2$ and $\lambda = 4$.

When $\lambda = 2$ the eigenvector is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

When $\lambda = 4$ the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Hence the solution is $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$.

(h) $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ has eigenvalues $\lambda = 1$ twice.

When $\lambda = 1$ the eigenvector is $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

To find the second solution we need to find a vector $\mathbf{b}$ such that $(A - \lambda I)\mathbf{b} = \mathbf{a}$. Writing $\mathbf{b} = \begin{pmatrix} c \\ d \end{pmatrix}$ we want to solve

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

One solution to this is $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Note this solution is not unique and we can add on to it any multiple of $\mathbf{a}$.

Using $\mathbf{a}$ and $\mathbf{b}$ found above the solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right\}.$$