Question

Express the following system of first order differential equations in matrix form, and find the eigenvalues and eigenvectors if the associated coefficient matrix:

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = 2x + y.$$

Hence find the general solution to the equations.

Answer

In matrix form: $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} &= (2-\lambda)(1-\lambda) - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1) = 0 \\ \text{so} \quad \lambda &= -1, 4 \end{aligned}$$

$\underline{\lambda = -1}$ Solve $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\left.\begin{aligned} 3x + 3y &= 0 \\ 2x + 2y &= 0 \end{aligned}\right\}$ $\begin{aligned} \text{let} \quad x &= \alpha \\ \text{so} \quad y &= -\alpha \end{aligned}$

Suitable eigenvector $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$

$\underline{\lambda = 4}$ Solve $\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\left.\begin{aligned} -2x + 3y &= 0 \\ 2x - 3y &= 0 \end{aligned}\right\}$ $\begin{aligned} \text{let} \quad x &= 3\beta \\ \text{so} \quad y &= 2\beta \end{aligned}$

Suitable eigenvector $\begin{pmatrix} 3\beta \\ 2\beta \end{pmatrix}$

General solution to equations:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} e^{-t} + \begin{pmatrix} 3\beta \\ 2\beta \end{pmatrix} e^{4t} \\ &= \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \beta \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} \end{aligned}$$