QUESTION There are certain matrices (in particular the Jacobian, Hessian and Wronksian) the elements of which consist of functions and/or their derivatives.
Let $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where each of the coordinate functions u_r is a function of all the variables x_s. The *Jacobian matrix* D has $(D)_{rs} = \partial u_r / \partial x_s$.

The *Jacobian* (or *Jacobian determinant*) is the determinant of this matrix. For example, if $m = n = 2$ then the Jacobian is denoted by $\partial(u_1, u_2)/\partial(x_1, x_2)$ and in the case when

$$\begin{aligned} u_1 &= x_1 + x_2, \\ u_2 &= x_1 x_2^2, \end{aligned}$$

then

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ x_2^2 & 2x_1x_2 \end{vmatrix} = 2x_1x_2 - x_2^2.$$

The *Hessian matrix* H is defined when $m = 1$ and has $(H)_{rs} = \partial^2 u/\partial x_r \partial x_s$; the *Hessian* (or *Hessian determinant*) is $\det H$. For example, if $u = x^2y^2z^2$ then

$$H = \begin{bmatrix} \partial^2 u/\partial x^2 & \partial^2 u/\partial x \partial y & \partial^2 u/\partial x \partial z \\ \partial^2 u/\partial y \partial x & \partial^2 u/\partial y^2 & \partial^2 u/\partial y \partial z \\ \partial^2 u/\partial z \partial x & \partial^2 u/\partial z \partial y & \partial^2 u/\partial z^2 \end{bmatrix} = \begin{bmatrix} 2y^2z^2 & 4xyz^2 & 4xy^2z \\ 4xyz^2 & 2x^2z^2 & 4x^2yz \\ 4xy^2z & 4x^2yz & 2x^2y^2 \end{bmatrix}.$$

[For everyday functions, the mixed partial derivatives are equal, in which case H is a symmetric matrix.]

(a) Find the Jacobian matrix and the Jacobian for the following set of functions:

$$\begin{aligned} u &= x^2 + y^2 + z^2, \\ v &= xy + yz + zx, \\ w &= x + y + z. \end{aligned}$$

(b) Find the Hessian matrix and Hessian of $u = ax^3 + 3bx^2y + 3cxy^2 + dy^3$.
ANSWER

(a)

$$J = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{bmatrix}$$

The determinant=0. This can be proved in various ways, e.g.
$\frac{1}{2}$row 1+row 2=$(x + y + z)$row 3.

(b)

$$H = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial u} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6ax + 6by & 6bx + 6cy \\ 6bx + 6cy & 6cx + 6dy \end{bmatrix}$$

$$\begin{aligned} \det H &= 36\{(ax + by)(cx + dy) - (bx + cy)^2\} \\ &= 36\{(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2\} \end{aligned}$$