QUESTION **Test 1** The characteristic equation of any matrix A can be written

$$\lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} - \ldots + (-1)^n c_n = 0. \tag{1}$$

If all c_r are positive then all solutions λ are positive.

(This is easy to prove: if $\lambda = \alpha < 0$ is a root then substituting $\alpha = -\beta$ (where $\beta > 0$) into (1) gives $(-1)^n \times$ a sum of strictly positive terms, so this cannot be zero.)

Definition If A is an $n \times n$ matrix then the following submatrices are called its *principal submatrices*:

$$\begin{bmatrix} a_{11} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \ \textit{etc.}$$

Test 2 A symmetric matrix A is positive definite if and only if each of its principal submatrices (including A itself) has positive determinant.

Use each of the two tests above to show that one of the following matrices is positive definite but the other is not. To what type of form does the other matrix correspond?

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

For each of the following quadratic forms, use eigenvalues and eigenvectors to rotate the axes in order to identify what type of conic it represents.

(a) $9x^2 - 4xy + 6y^2 - 10x - 20y - 5 = 0$,
(b) $3x^2 - 8xy - 12y^2 - 30x - 64y = 0$,
(c) $4x^2 - 20xy + 25y^2 - 15x - 6y = 0$,
(d) $9x^2 + 12xy + 4y^2 - 52 = 0$.

ANSWER $-\lambda^3+8\lambda^2-19\lambda+12 = 0$ has coefficients which alternate in sign; the principal subdeterminants are 3,5 and 12. So the quadratic form is positive definite. $-\lambda^3 + 5\lambda 62 - \lambda - 7 = 0$ does not have coefficients which alternate in sign; the principal subdeterminants are 1,-3 and -7. So the quadratic form is not positive definite, it is indefinite.