Question

Show that the formula of real analysis Log(ab) = Log(a) + Log(b) where $a,\ b \notin C$, does not always hold in the full complex plane, i.e., when $a,\ b \in C$. However, show that Log(ab) = Log(a) + Log(b) + 2nπi where n is an integer.

Answer

$$\begin{aligned}\mathrm{Log} &= \log|ab| + i\mathrm{Arg(ab)} \\ &= \log|a| + \log|b| + i\mathrm{Arg(ab)}\end{aligned}$$

Now if Arg(ab) = Arg(a) + Log(b) ($\star$) we have

$$\begin{aligned}\mathrm{Log(ab)} &= \log|a| + i\mathrm{Arg(a)} + \log|\mathrm{b}| + \mathrm{iArg(b)} \\ &= \mathrm{Log(a)} + \mathrm{Log(b)}\end{aligned}$$

But this only works if ($\star$) holds.

What happens if $\left\{\begin{array}{l}\mathrm{Arg(a)} = \dfrac{3\pi}{4} \\ \mathrm{Arg(b)} = \dfrac{3\pi}{4}\end{array}\right\}$ say.

$$\mathrm{Arg(a)} + \mathrm{Arg(b)} = \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{3\pi}{2}$$

But $\dfrac{3\pi}{2}$ doesn't lie in the range on <u>Arg</u>

since $-\pi < \mathrm{Arg} \le \pi$ and $\pi < \dfrac{3\pi}{2}$, so here we need

$$\mathrm{Arg(ab)} = \mathrm{Arg(a)} + \mathrm{Arg(b)} - \underbrace{2\pi}_{}$$

to get the RHS back into $(-\pi, \pi]$

so in general

$$\mathrm{Arg(ab)} = \mathrm{Arg(a)} + \mathrm{Arg(b)} + 2\mathrm{n}\pi\mathrm{i}$$

n integer ($>$ or $<$ 0)

NB for Arg(a) = Arg(b) = 0 Log(ab) = Log(a) + Log(b) the usual <u>real</u> $a,\ b$ formula.