Question

Find the general solution for the differential equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = t + \cos t$$

Answer

Need general solution of

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = t + \cos t = L[x]$$

Complementary Function: $m^2 - 2m + 2 = 0 \Rightarrow (m-1)^2 = -1 \Rightarrow m = -1 \pm j$
$\Rightarrow x(t) = e^t(A\cos t + B\sin t)$
For the particular integral to $L[x_1^*] = t$ try

$$\begin{aligned} x_1^* &= a_1 t + a_0 \Rightarrow \frac{dx_1^*}{dt} = a_1 \Rightarrow \frac{d^2x_1^*}{dt^2} = 0 \\ L[x_1^*] &= 0 - 2a_1 + 2(a_1 + a_0) \\ &= (2a_0 - 2a_1) + 2a_1 t \\ &\equiv t \end{aligned}$$

Solving gives $a_1 = \frac{1}{2}$, $a_0 = \frac{1}{2} \Rightarrow x_1^* = \frac{1}{2}(t+1)$

For the particular integral to $L[x_2^*] = \cos t$ try

$$\begin{aligned} x_2^* &= b_1\cos t + b_2\sin t \\ \frac{dx_2^*}{dt} &= b_2\cos t - b_1\sin t \\ \frac{d^2x_2^*}{dt^2} &= -b_1\cos t - b_2\sin t \\ L[x_2^*] &= -b_1\cos t - b_2\sin t - 2(b_2\cos t - b_1\sin t) + 2(b_1\cos t + b_2\sin t) \\ &= \cos t[-b_1 - 2b_2 + 2b_1] + \sin t[-b_2 + 2b_1 + 2b_2] \\ &= \cos t[b_1 - 2b_2] + \sin t[b_2 + 2b_1] \\ &\equiv \cos t + 0 \times \sin t \qquad \text{by hypothesis} \end{aligned}$$

Hence $\begin{aligned} b_1 - 2b_2 &= 1 \\ b_2 + 2b_1 &= 0 \end{aligned} \Rightarrow b_1 = \frac{1}{5} \;\; b_2 = -\frac{2}{5}$
Hence $x_2^* = \frac{1}{5}\cos t - \frac{2}{5}\sin t$
The General Solution is:

$$\begin{aligned} x &= x_c + x_1^* + x_2^* \\ &= e^t(A\cos t + B\sin t) + \frac{1}{2}(t+1) + \frac{1}{5}\cos t - \frac{2}{5}\sin t \end{aligned}$$