Vector Functions and Curves
One variable functions

Question
Show that $\underline{r} = \underline{r}_0 \cos(\omega t) + (\underline{v}_0/\omega)\sin(\omega t)$ satisfies the initial-value problem.

$$\begin{aligned} \frac{d^2\underline{r}}{dt^2} &= -\omega^2 \underline{r} \\ \underline{r}'(0) &= \underline{v}_0 \\ \underline{r}(0) &= \underline{r}_0 \end{aligned}$$

In this case, describe the path $\underline{r}(t)$ and determine what happens to the paths if $\underline{r}_0$ is perpendicular to $\underline{v}_0$.
Answer

$$\begin{aligned} \underline{r} &= \underline{r}_0 \cos\omega t + \left(\frac{\underline{v}_0}{\omega}\right)\sin\omega t \\ \Rightarrow \frac{d\underline{r}}{dt} &= -\omega \underline{r}_0 \sin\omega t + \underline{v}_0 \cos\omega t \\ \Rightarrow \frac{d^2\underline{r}}{dt^2} &= -\omega^2 \underline{r}_0 \cos\omega t - \omega \underline{v}_0 \sin\omega t \\ &= -\omega^2 \underline{r} \end{aligned}$$

$$\underline{r}(0) = \underline{r}_0, \qquad \left.\frac{d\underline{r}}{dt}\right|_{t=0} = \underline{v}_0$$

See that $\underline{r} \bullet (\underline{r}_0 \times \underline{v}_0)$ for all t.
So the path lies in a plane through the origin with normal

$$\underline{N} = \underline{r}_0 \times \underline{v}_0.$$

So choose the system of coordinates so that

$$\begin{aligned} \underline{r}_0 &= a\underline{i} \qquad (a > 0) \\ \underline{v}_0 &= \omega b\underline{i} + \omega c\underline{j} \qquad (c > 0) \end{aligned}$$

$\Rightarrow \underline{N}$ is in the direction of $\underline{k}$.
Parameterization of the path gives

$$\begin{aligned} x &= a\cos\omega t + b\sin\omega t \\ y &= c\sin\omega t \end{aligned}$$

The curve has the quadratic equation

$$\frac{1}{a^2}\left(x - \frac{by}{c}\right)^2 + \frac{y^2}{c^2} = 1$$

so it is a conic section. As the path is bounded by

$$|\underline{r}(t)| \leq |\underline{r}_0| + (|\underline{v}_0|/\omega)$$

it must be an ellipse.
If $\underline{r}_0$ is perpendicular to $\underline{v}_0$, then $b = 0$, making the path an ellipse with equation

$$(x/a)^2 + (y/c)^2 = 1$$

and semi-axes $a = |\underline{r}_0|$ and $c = |\underline{v}_0|/\omega$.