Vector Functions and Curves
Applications

Question
If an object moves with position vector $\underline{r}(t)$ satisfying

$$\frac{d\underline{r}}{dt} = \underline{a} \times (\underline{r}(t) - \underline{b})$$

and with $\underline{r}(0) = \underline{r}_0$.
$\underline{a}$, $\underline{b}$ and $\underline{r}_0$ are constant vectors with $\underline{a} \neq 0$.
Describe the path along which this object moves.
Answer

$$\begin{aligned} \frac{d}{dt}|\underline{r} - \underline{b}|^2 &= 2(\underline{r} - \underline{b}) \bullet \frac{d\underline{r}}{dt} \\ &= 2(\underline{r} - \underline{b}) \bullet (\underline{a} \times (\underline{r} - \underline{b})) \\ &= 0 \end{aligned}$$

$\Rightarrow |\underline{r} - \underline{b}|$ is constant.
For all t, the object will lie on the sphere with radius $\underline{r}_0 - \underline{b}$ and centered at the point with position vector $\underline{b}$.

$$\frac{d}{dt}(\underline{r} - \underline{r}_0) \bullet \underline{a} = (\underline{a} \times (\underline{r} - \underline{b})) \bullet \underline{a} = 0$$

$\Rightarrow \underline{r} - \underline{r}_) \perp \underline{a}$
For all t, the object will lie on the plane which passes through $\underline{r}_0$ and has normal $\underline{a}$.
So the path of the object is the intersection of the plane and the sphere. I.e. it lies on the circle of intersection between the sphere and the plane.
Thus the angle between $\underline{r} - \underline{b}$ and $\underline{a}$ must be constant.
$\Rightarrow \left|\frac{d\underline{r}}{dt}\right|$ is constant.
So the path is the entire circle of intersection.