QUESTION

(a) For the (zero-one) knapsack problem

$$\begin{array}{ll} \text{maximize} & z = \sum_{i=1}^{n} c_i x_i \\ \text{subject to} & \sum_{i=1}^{n} a_i x_i \le b \\ & x_i = 0 \text{ or } 1 \text{ for } i = 1, \ldots, n \end{array}$$

where a_i and c_i are positive for $i = 1, \ldots, n$, and $\sum_{i=1}^{n} a_i > b$, describe a procedure for solving the linear programming relaxation. Also, prove that your procedure provides an optimal solution of the linear programming relaxation.

(b) Use a branch and bound algorithm to solve the following (zero-one) knapsack problem. In your algorithm, always choose a node of the search tree with the largest upper bound to be explored next.

$$\begin{array}{ll} \text{Maximize} & z = 18x_1 + 25x_2 + 15x_3 + 8x_4 + 11x_5 + 4x_6 + 2x_7 \\ \text{subject to} & 9x_1 + 13x_2 + 8x_3 + 5x_4 + 7x_5 + 5x_6 + 3x_7 \le 28 \\ & x_i = 0 \text{ or } 1 \text{ for } i = 1, \ldots, 7 \end{array}$$

ANSWER

(a) Index the variables so that $\frac{c_1}{a_1} \ge \ldots \ge \frac{c_n}{a_n}$. Find index k such that $\sum_{i=1}^{k} a_i < b \le \sum_{i=1}^{k+1} a_i$.

The solution of the linear programming relaxation is

$$\begin{aligned} x_i &= 1 \text{ for } i = 1, \ldots, k \\ x_{k+1} &= \frac{\left(b - \sum_{i=1}^{k} a_i\right)}{a_{k+1}} \\ x_i &= 0 \text{ for } i = k+2, \ldots n \end{aligned}$$

The solution value is

$$z = \sum_{i=1}^{k} c_i + c_{k+1} \frac{\left(b - \sum_{i=1}^{k}\right)}{a_{k+1}}$$

The dual problem is

Minimize	$z_D = b_y + \sum_{i=1}^n z_i$
subject to	$y \geq 0,\ z_i \geq 0\ i = 1, \ldots n$
	$a_i y + z_i \geq c_i\ i = 1, \ldots, n$

Consider the dual solution

$$\begin{aligned} y &= \frac{c_{k+1}}{a_{k+1}} \\ z_i &= c_i - \frac{a_i c_{k+1}}{a_{k+1}}\ i = 1, \ldots, k \end{aligned}$$

$z_i = 0\ i = k+1, \ldots, n$

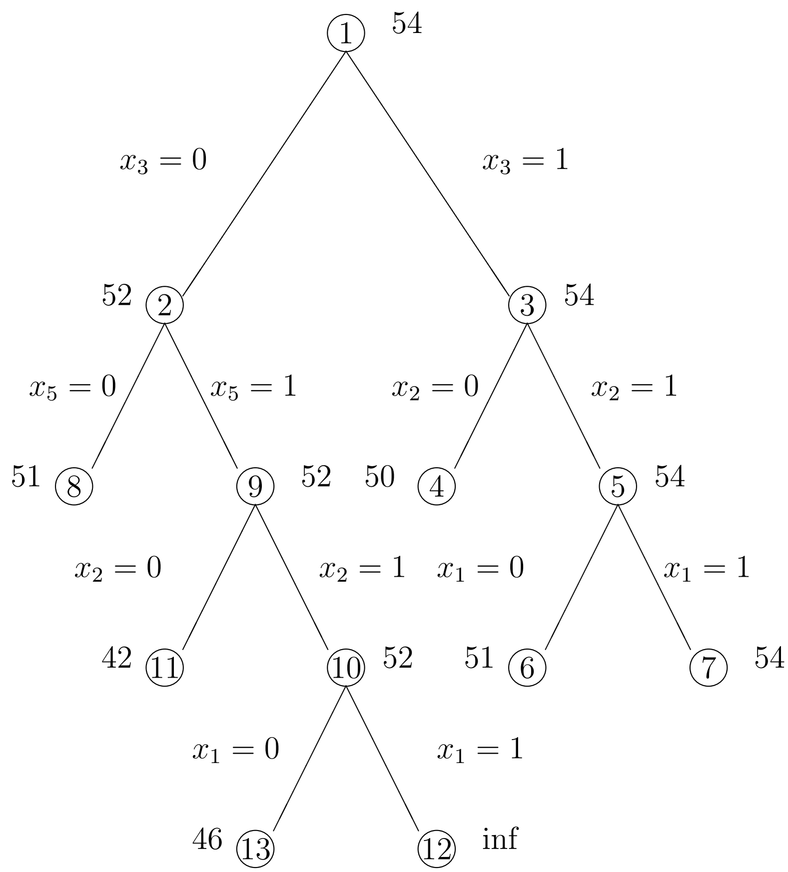
Note that $z_i \geq 0$ by the indexing of the variables

$$\begin{aligned} a_i y + z_i &= c_i \text{ for } i = 1, \ldots k \\ a_i y + z_i &= a_i \frac{c_{k+1}}{a_{k+1}} \geq c_i \text{ for } i = k+1, \ldots, n \end{aligned}$$

so the solution is feasible.

$$z_D = b\frac{c_{k+1}}{a_{k+1}} + \sum_{i=1}^{k} c_i - \frac{c_{k+1}}{a_{k+1}} \sum_{i=1}^{k} a_i = z$$

Therefore the primal and dual solutions are optimal.


1 54
$x_3 = 0$
$x_3 = 1$
52 2
3 54
$x_5 = 0$
$x_5 = 1$
$x_2 = 0$
$x_2 = 1$
51 8
9 52
50 4
5 54
$x_2 = 0$
$x_2 = 1$
$x_1 = 0$
$x_1 = 1$
42 11
10 52
51 6
7 54
$x_1 = 0$
$x_1 = 1$
46 13
12 inf

(b)

Node 1	$UB = 18 + 25 + \lfloor \frac{3}{4} 15 \rfloor = 54$
	$LB = 43$
Node 2	$UB = 18 + 25 + 8 + \lfloor 11\frac{1}{7} \rfloor = 52$
	$LB = 51$
Node 3	$UB = 18 + \lfloor \frac{11}{13} 25 \rfloor + 15 = 54$
	$LB = 33$
Node 4	$UB = 18 + 8 + \lfloor \frac{6}{7} 11 \rfloor + 15 = 50$
	$LB = 41$
Node 5	$UB = 18 + 8 + \lfloor \frac{7}{9} 18 \rfloor + 15 = 50$
	$LB = 41$
Node 6	$UB = 8 + \lfloor \frac{2}{7} 11 \rfloor + 25 + 15 = 51$
	$LB = 48$
Node 8	$UB = 18 + 25 + 8 + \lfloor \frac{1}{5} 4 \rfloor = 51$
	$LB = 51$
Node 9	$UB = 18 + \lfloor \frac{12}{13} 25 \rfloor + 11 = 52$
	$LB = 29$
Node 10	$UB = 18 + 8 + 4 + \lfloor \frac{2}{3} 2 \rfloor + 11 = 42$
	$LB = 41$
Node 11	$UB = \lfloor \frac{8}{9} 18 \rfloor + 25 + 11 = 52$
	$LB = 36$
Node 12	$UB = 8 + \lfloor \frac{3}{5} 4 \rfloor + 25 + 11 = 46$
	$LB = 44$

Optimal solution $x_1 = x_2 = x_4 = 1$ $x_3 = x_5 = x_6 = 0$ $z + 51$