QUESTION

(a) Define the following terms

- **(i)** direct product,
- **(ii)** isomorphism,
- **(iii)** normal subgroup.

(b) Show that the kernel of a homomorphism is a normal subgroup (you may assume that it is a subgroup).

(c) Let G be a group with identity element e and let H and K be subgroups of G with $H \cap K = \{e\}$. Show that if $hk = kh$ for any $h \in H$ and any $k \in K$ then the function $f : H \times K \longrightarrow G$ given $f(h,k) = hk$ is an injective homomorphism. Show that if G is a group in which every element has order 2 then G is abelian, and deduce that any two non-identity elements of G generate a subgroup isomorphic to the Klein 4-group.

Give an example to show that an abelian group can contain two elements of order 3 without containing a subgroup isomorphic to $Z_3 \times Z_3$.

ANSWER

(a) **(i)** $(G,*), (H,.)$ are groups.

$\{(g,h)|g \in G, h \in H\} = G \times H$ with $(g_1,h_1)(g_2,h_2) = (g_1 * g_2, h_1.h_2)$ is the direct product.

(ii) An isomorphism is a bijective function $f : G \longrightarrow H$ with $f(g*k) = f(g).f(k) \forall g,k \in G$.

(iii) A subgroup $H < G$ is normal if $g^{-1}Hg = H \forall g \in G$.

(b)

$$\begin{aligned} f(g^{-1}kg) &= f(g^{-1}f(k)f(g) \forall g \in G, k \in \text{kernel} \\ &= f(g^{-1})e_H f(g) \\ &= f(g^{-1})f(g) \\ &= f(g^{-1}g) = f(e_G) = e_H \end{aligned}$$

(c) $f(h,k) = e \Leftrightarrow hk = e \Leftrightarrow h = k^{-1}$. But $h = k^{-1} \Rightarrow h \in H \cap K = \{e\}$ so $h = e$.

Similarly $k = e$ and $\text{Ker}(f) = \{(e,e)\}$ and f is injective.

$$\begin{aligned} f((h_1,k_1)(h_2,k_2)) &= f(h_1h_2,k_1k_2) \\ &= h_1h_2k_1k_2 \\ &= h_1k_1h_2k_2 \\ &= f(h_1,k_1)f(h_2,k_2) \end{aligned}$$

If every element in G has order 2 then $(gh)^2 = e \forall g,h \in G$ and $g = g^{-1},\ h = h^{-1}$ so $e = (gh)^2 = ghgh = ghg^{-1}h^{-1} \Rightarrow gh = hg \forall g,h \in G$.

Now $\langle g,h\rangle = \langle g\rangle \times \langle h\rangle$ since the map $f : \langle g \times\rangle \longrightarrow G$ is an isomorphism onto its image.

C_3 contains 2 elements of order 2 but is not isomorphic to $c_3 \times C_3$