QUESTION

(a) The element σ is an element of the finite permutation group S_n. Explain the relationship between the cycle structure if σ and its order, and use this to find the smallest positive integer n such that S_n contains an element of order 12.

List the possible cycle structures for elements of order 14 in S_9 and use this to find the number of such elements. (You are NOT required to list them all.)

(b) Express the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 7 & 6 & 3 & 5 & 1 & 9 & 8 \end{pmatrix}$ in disjoint cycle notation and as a product of transpositions. Fine the order and sign of σ and calculate the order of σ^{2000}.

(c) Say what it means for a permutation in the symmetric group S_n to be even, and show that a permutation is even if and only if it can be written as a product of 3-cycles.

ANSWER

(a) The order of σ is the least common multiple of the lengths of its cycles. Ignoring 1-cycles the possible cycle structures for an element of order 12 are $[12]$, $[3, 4]$ and the smallest n such that S_n contains an element of order 12 is 7.

$[2, 7]$ is the only possible cycle structure. There are $\frac{9.8}{2}$ possible transpositions and 6! different 7- cycles so $36.720 = 25,920$ different elements of order 14.

(b) σ =(1 2 4 6 5 3 7)(8 9)=(1 2)(2 4)(4 6)(6 5)(5 3)(3 7)(8 9) which has order 14 and sign -1.

$2000 = (14.142) + 12$ so $\sigma^{2000} = \sigma^{12}$ and σ^{12} has order 7.

(c) A permutation is even $\Leftrightarrow$ it can be written as a product of an even number of transpositions.

Any 3-cycle $(x\ y\ z)$ can be written as $(x\ y\ z) = (x\ y)(y\ z)$ so any product of 3-cycles is even.

Any pair of transpositions can be written as a 3-cycle and any poair $x\ y)(u\ v)$ can be written as the product $(x\ y)(y\ u)(y\ u)(u\ v)$ so as the product $(x\ y\ u)(y\ u\ v)$.