Question

(a) Show that all the roots of the equation

$$(1+x)^{2n+1} = (1-x)^{2n+1}$$

are given by

$$\pm i \tan\left(\frac{k\pi}{2n+1}\right) \qquad k = 0, 1, 2, \cdots, n$$

By putting $n = 2$ show that

$$\tan^2\left(\frac{\pi}{5}\right)\tan^2\left(\frac{2\pi}{5}\right) = 5.$$

(b) Let $z = x + iy$ and $w = u + iv$. If $w = z^2 + 2z$ show that the line $v = 2$ is the image of a rectangular hyperbola in the z-plane. Sketch this hyperbola.

Answer

(a)

$$\begin{aligned}
(1+x)^{2n+1} &= (1-x)^{2n+1} \\
\text{So} \quad \frac{1+x}{1-x} &= e^{\frac{2\pi i}{2n+1}k} \\
x &= \frac{e^{\frac{2\pi i}{2n+1}k} - 1}{e^{\frac{2\pi i}{2n+1}k} + 1} \\
&= \frac{e^{\frac{\pi i k}{2n+1}} - e^{-\frac{\pi i k}{2n+1}}}{e^{\frac{\pi i k}{2n+1}} + e^{-\frac{\pi i k}{2n+1}}} \\
&= i \tan \frac{\pi k}{2n+1} \qquad k = -n, \cdots, n \\
&= \pm i \tan \frac{\pi k}{2n+1} \qquad k = 0, \cdots, n
\end{aligned}$$

Putting $n = 2$. The equation reduces to $x(x^4 + 10x^2 + 5) = 0$.

So the product of the non-zero roots is 5.

i.e. $\tan^2\left(\frac{\pi}{5}\right)\tan^2\left(\frac{2\pi}{5}\right) = 5.$

(b) $z = x + iy$

$w = u + iv$

Therefore as $w = z^2 + 2z$, $u + iv = x^2 - y^2 + 2ixy + 2(x + iy)$

So $v = 2xy + 2x$

Thus $v = 2$ if and only if $2xy + 2x = 2$ and $y(x + 1) = 1$

This is a rectangular hyperbola.