Question

Solve the following Bernoulli equations

1. $$x\frac{dy}{dx} = y - \frac{1}{y} \qquad (*)$$

2. $$3\frac{dy}{dx} + xy = \frac{x}{y^2} \qquad (*)$$

Answer

a) $x\frac{dy}{dx} - y = -\frac{1}{y}$ is a Bernoulli equation with $n = -1$, so put $y = u^{\frac{1}{1-n}} = u^{\frac{1}{2}} \Rightarrow$ either $u^{\frac{-1}{2}} = 0$ or $\frac{1}{2}x\frac{du}{dx} - u = -1$

$\frac{du}{dx} - \frac{2}{x}u = -\frac{2}{x} \Rightarrow I(x) = exp\left(\int -\frac{2}{x}dx\right) = exp(-2\ln x) = \frac{1}{x^2}$

$\frac{d}{dx}\left(\frac{1}{x^2}u\right) = -\frac{2}{x^3} \Rightarrow \frac{1}{x^2}u = \frac{1}{x^2} + A \Rightarrow u = 1 + Ax^2$

$y = \left(1 + Ax^2\right)^{\frac{1}{2}}$

b) $3\frac{dy}{dx} + xy = \frac{x}{y^2}$ is a Bernoulli equation with $y = -2$

put $y = u^{\frac{1}{1-n}} = u^{\frac{1}{3}} \Rightarrow$ either $u^{\frac{-2}{3}} = 0$ or $\frac{du}{dx} + xu = x$

So $I(x) = exp\left(\int xdx\right) = e^{\frac{1}{2}x^2}$

$\frac{d}{dx}\left(e^{\frac{1}{2}x^2}u\right) = xe^{\frac{1}{2}x^2} \Rightarrow e^{\frac{1}{2}x^2}u = e^{\frac{1}{2}x^2} + A$

$u = 1 + Ae^{-\frac{1}{2}x^2} \Rightarrow y = \left(1 + Ae^{-\frac{1}{2}x^2}\right)^{\frac{1}{3}}$