QUESTION

An integer n is called 3-perfect if $\sigma(n) = 3n$. Show that 120 is 3-perfect. Find all 3-perfect numbers n of the form $2^k.3.p$ where p is an odd prime.

[Hints:

(i) Treat the case $p = 3$ separately, so that you can multiplicity to calculate $\sigma(n)$.

(ii) Evaluate $\sigma(n)$ and use the equation $\sigma(n) = 3n$ to get an expression for 2^k in terms of p. Hence deduce $p \leq 8$.

(iii) Deal with the possible values of p separately.]

ANSWER

$\sigma(120) = \sigma(2^3, 3, 5) = \frac{(2^4-1)}{(2-1)}.\frac{(3^2-1)}{(3-1)}.\frac{(5^2-1)}{(5-1)} = 15.\frac{8}{2}.\frac{24}{4} = 15.4.6 = 2^3.3^2.5 = 3.(2^3.3.5) = 3.120.$

Thus 120 is 3-perfect.

Now suppose $n = 2^k.3.p$ is 3-perfect.

CASE 1: $p = 3$, so $n = 2^k.3^2$ and $\sigma(n) = \frac{2^{k+1}-1}{2-1}.\frac{3^3-1}{3-1} = (2^{k+1} - 1).\frac{26}{2} = 13.(2^{k+1} - 1)$. As $\sigma(n) = 3n$, we have $13(2^{k+1} - 1) = 3.2^k.3^2 = 2^k.3^3$ which is clearly impossible, as the prime 13 divides the left-hand side but not the right.

CASE 2: $p \neq 3$, so $n = 2^k.3.p$, and $\sigma(n) = \frac{2^{k+1}-1}{2-1}.\frac{3^2-1}{3-1}.\frac{p^2-1}{p-1} = (2^{k+1}-1).4.(p+1)$ (using $p^2 - 1 = (p-1)(p+1)$.) AS $\sigma(n) = 3n$, $(2^{k+1} - 1).4.(p+1) = 2^k.3^2.p$. Thus $2^k(3^2p - 2.4.(p+1)) = -4(p+1)$ (using $2^{k+1} = 2.2^k$), i.e. $2^k(p-8) = -4(p+1)$. As the right-hand side of this equation is negative, so is the left, so $p < 8$. As we know p is odd, and $p \neq 3$, only $p = 5$ and $p = 7$ are possible.

If $p = 5$. the equation $2^k(p-8) = -4(p+1)$ gives $2^k(-3) = -4.6$, giving $k = 3$ and $n = 2^3.3.5 = 120$ (the case we've already dealt with.)

If $p = 7$, the equation $2^k(p-8) = -4(p+1)$ gives $2^k(-1) = -4.8$, giving $k = 5$. Thus $n = 2^5.3.7 = 671$, so 120 and 672 are the only 3-perfect numbers of the form $2^k.3.p$ where p is an odd prime.