QUESTION

(i) Prove that $d(n)$ is odd if and only if a is the square of some integer m.

(ii) Prove that if $n > 1$, then $\sigma(n) \geq n+1$, with equality holding if and only if n is prime.

(iii) Prove $\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$ for every $n \in N$.

(iv) Describe all positive integers n which satisfy $d(n) = 12$. (There are infinitely many of them- you need to find a way to describe them all.

ANSWER

(i) If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ then $d(n) = (\alpha_1+1)(\alpha_2+1)\dots(\alpha_k+1)$. Thus $d(n)$ is odd $\Leftrightarrow$ each α_i is even $\Leftrightarrow$ n is a square.

(ii) $\sigma(n)$= sum of positive divisors of n.

Now 1 and n are both positive divisors of n, and are distinct as $n > 1$. Thus $\sigma(n) \geq 1+n$, and equality holds if and only if there are no other positive divisors of n, i.e.$\Leftrightarrow$ n is prime.

(iii) Consider $n. \sum_{d|n} \frac{1}{d} = \sum_{d|n} \frac{n}{d}$.

As d varies through the divisors of n, so does $\frac{n}{d}$. Thus the sum $\sum_{d|n} \frac{n}{d}$ is just the sum of the positive divisors of n, so it can be rewritten as $\sum_{d|n} d = \sigma(n)$. Hence $n \sum_{d|n} \frac{1}{d} = \sigma(n)$, so $\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$.

(iv) Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, with each $\alpha_i \geq 1$. Then $d(n) = (\alpha_1 + 1)(\alpha_2+1)\dots(\alpha_k+1) = 12$.

Now each $\alpha_i + 1$ is ≥ 2, so we wish to find in how many ways we can write 12 as a product of k factors each ≥ 2. Now $12 = 2^2.3$, so certainly there can't be more than 3 factors, so $k = 1, 2$ or 3.

If $k = 1$, then $(\alpha_1 + 1) = 12$, so $\alpha_1 = 11$, and so all n of the form p^{11} satisfy $d(n) = 12$.

If $k = 2$, then $(\alpha_1+1)(\alpha_2+1) = 12$. If we allow the primes p_I to take any values, we may assume that they are arranged in order of increasing α_i, so we may as well assume $|alpha_1 \leq \alpha_2$. Thus the possible solutions are $\alpha_1 = 1,\ \alpha_2 = 5$ and $\alpha_1 = 2,\ \alpha_2 = 3$ and the possible forms of integers n given by these are pq^5 and p^2q^3(p and q distinct primes).

If $k = 3$, then $(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) = 12$, and taking the p_i to be arranged so that the α_I increase, as above, the only possibility is

$\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 2$, so the relevant n's are pqr^2, where p, q and r are distinct primes.

Thus our final list, with $p,\ q$ and r any distinct primes is p^{11}, pq^5, p^2q^3 and pqr^2.