QUESTION Show that $u(x,y) = 2x - x^3 + 3xy^2$ is harmonic. Find a conjugate harmonic function $v(x,y)$ and identify the corresponding analytic function $u + iv$.

ANSWER Easy to show $u_{xx} + u_{yy} = 0$, so u is harmonic. Let v be the conjugate harmonic function. Then

$$v_y = u_x = 2 - 3x^2 + 3y^2$$

Thus

$$v(x,y) = 2y - 3x^2y + y^3 + \phi(x).$$

Now $v_x = -6xy + \phi'(x) = -u_y = -6xy$. Thus $\phi(x) =$constant, so

$$v(x,y) = 2y - 3x^2y + y^3 + \text{constant}$$

and $u + iv = 2z - z^3 + ic$, where c is a real constant. (Note that we have expressed this in terms of $z = x + iy$.)