QUESTION In each of the following case state whether or not the function f of the complex variable $z = x + iy$ is (i) continuous, (ii) differentiable at $z = 0$ and give a brief justification for each answer.

(a) $f(z) = |z|$,

(b) $f(z) = xy$

(c) $f(z) = \frac{x^2-y^2}{x^2+y^2}$ for $z \neq 0$ and $f(0) = 1$.

(d) $f(z) = \frac{\sinh z}{z}$, $z \neq 0$ and $f(0) = 1$.

ANSWER

(a) Continuous at 0 as $\lim_{z\to 0} |z| = 0 (= |0|)$. Not differentiable at 0 as $\lim_{h\to 0} \frac{|h|-|0|}{h} = \frac{|h|}{h}$ and this means that the limit does not exist as it depends on the path we use to reach zero.

(b) Continuous at 0(same reasons as in (a).) Is is differentiable at 0 as the Cauchy-Riemann equations hold at 0, and the partial derivatives exist in a neighbourhood of 0 and are continuous there. (Theorem 3.3)

(c) Change to polar coordinates, $x = r\cos\theta, y = r\sin\theta$. Then $f(z) = \cos 2\theta$. Hence not continuous and hence not differentiable at 0.

(d) It is easy to see that L'hôpital's rule applies in complex variables as the same proof as in real variables holds and this shows that f is continuous at). It is differentiable at 0 for reasons that will be apparent later in the course. However, you could just use the definition and then we have to examine $\lim_{z\to 0} \frac{f(z)-f(0)}{z}$. This is equal to $\lim_{z\to 0} \frac{\sinh z - z}{z^2}$, and you can find this (and prove it exists) by using L'Hôpital twice. You also need to show that the partial derivatives of f exist in a neighbourhood of 0 and are continuous there.