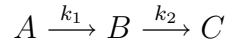


### Question

Consider a sequential reaction



in which atoms of type  $A$  decay into atoms of type  $B$ , and atoms of type  $B$  decay into atoms of type  $C$ , which are stable. Let  $a(t), b(t), c(t)$  be the number of atoms at time  $t$  of type  $A, B, C$  respectively. The numbers of atoms are large, so that they may be taken to vary continuously, and the processes may then be modelled by a pair of coupled first order differential equations describing the rates of the decay:

$$\frac{da}{dt} = -k_1 a, \quad \frac{db}{dt} = k_1 a - k_2 b,$$

together with the information that  $a(t) + b(t) + c(t)$  is constant. Assuming that at time  $t = 0$ ,  $a(0) = \alpha$  and  $b(0) = 0 = c(0)$  find expressions for the number of atoms of each type which are present at any given time.

### Answer

Number of atoms of type A:

$$\text{Solve } \frac{\partial a}{\partial t} = -K_1 a \text{ (Separation of variables)}$$

$$a(t) = \lambda e^{-K_1 t} (\lambda \text{ constant})$$

Using boundary condition,  $a(0) = \alpha$  gives  $\lambda = \alpha$

$$\text{so } \underline{a(t) = \alpha e^{-K_1 t}}$$

Number of atoms of type B:

$$\text{Solve } \frac{\partial b}{\partial t} = K_1 a - K_2 b = K_1 \alpha e^{-K_1 t} - K_2 b \text{ (from above)}$$

This gives a first order linear differential equation

$$\frac{\partial b}{\partial t} + K_2 b = K_1 \alpha e^{-K_1 t}$$

with integrating factor:  $I(t) = e^{K_2 \int dt} = e^{K_2 t}$

Multiplying equation by integrating factor gives

$$e^{K_2 t} \frac{\partial b}{\partial t} + K_2 b e^{K_2 t} = K_1 \alpha e^{(K_2 - K_1)t}$$

so

$$\frac{\partial}{\partial t} (b e^{K_2 t}) = K_1 \alpha e^{(K_2 - K_1)t}.$$

Integrating gives

$$be^{K_2t} = \frac{K_1\alpha}{K_2 - K_1}e^{(K_2-K_1)t} + D$$

where D is constant.

Boundary conditions  $b(0) = 0$  gives

$$0 = \frac{K_1\alpha}{K_2 - K_1} + D \Rightarrow D = -\frac{K_1\alpha}{K_2 - K_1}$$

Solution is

$$\underline{b(t) = \frac{K_1\alpha}{K_2 - K_1} (e^{-K_1t} - e^{-K_2t})}$$

Number of atoms of type C:  $c(t) = K - a(t) - b(t)$  for some constant  $K$ .  
Using boundary conditions  $a(0) = \alpha$ ,  $b(0) = 0 = c(0)$  we have

$$0 = K - \alpha - 0 \Rightarrow K = \alpha$$

so

$$\begin{aligned} c(t) &= \alpha - \alpha e^{-K_1t} - \frac{K_1\alpha}{K_2 - K_1} (e^{-K_1t} - e^{-K_2t}) \\ &= \underline{\alpha \left( 1 + \frac{K_2}{K_1 - K_2} e^{K_1t} - \frac{K_1}{K_1 - K_2} e^{-K_2t} \right)} \end{aligned}$$