Question

Solve the following differential equations, using the specified boundary conditions:

(a) $(x^2-3)\frac{dy}{dx}+2xy^2 = 0$, where $y = 1$ when $x = 2$. (Separation of Variables)

(b) $x\frac{dy}{dx} + y = \sin(x)$, where $y = 1$ when $x = \frac{\pi}{2}$. (Integrating Factor)

(c) $\frac{dy}{dx} + 2y = e^{3x}$, where $y = 0$ when $x = 0$. (Integrating Factor)

Answer

(a)

$$\begin{aligned}
(x^2-3)\frac{dy}{dx} &= -2xy^2 \\
\int \frac{dy}{y^2} &= -\int \frac{2x}{x^2-3}\,dx + C \\
-\frac{1}{y} &= -\ln|x^2-3| + C
\end{aligned}$$

General solution: $y = \dfrac{1}{\ln|x^2-3| - C}$

Boundary condition: $y = 1$ when $x = 2$

$$1 = \frac{1}{\ln(1) - C} = \frac{1}{-C} \Rightarrow C = -1$$

Particular solution: $y = \dfrac{1}{\ln|x^2-3| + 1}$

(b) Standard form: $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{\sin x}{x}$

Integrating factor: $I(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$

Multiply equation by integrating factor: $x\dfrac{dy}{dx} + y = \sin x$

so

$$\begin{aligned}
\frac{\partial}{\partial x}(xy) &= \sin x \\
xy &= \int \sin x\,dx + C
\end{aligned}$$

General solution: $y = -\frac{\cos x}{x} + \frac{C}{x}$

Boundary condition: $y = 1$ when $x = \frac{\pi}{2}$

$$1 = -\frac{0}{\frac{\pi}{2}} + \frac{2C}{\pi} \Rightarrow C = \frac{\pi}{2}$$

Particular solution: $y = -\frac{\cos x}{x} + \frac{\pi}{2x}$

(c) $\frac{dy}{dx} + 2y = e^{3x}$

Integrating factor: $I(x) = e^{2\int dx} = e^{2x}$

Multiply equation by integrating factor:

$$\begin{aligned} e^{2x}\frac{dy}{dx} + 2e^{2x}y &= e^{5x} \\ \frac{\partial}{\partial x}(e^{2x}y) &= e^{5x} \end{aligned}$$

so $e^{2x}y = \int e^{5x}\,dx + C$

General solution: $y = \frac{1}{5}e^{3x} + Ce^{-2x}$

Boundary condition: $y = 0$ when $x = 0$

so $0 = \frac{1}{5} + C \Rightarrow C = -\frac{1}{5}$

Particular solution: $y = \frac{1}{5}e^{3x} - \frac{1}{5}e^{-2x}$