Question

Sketch the *singular set* Σ and the *discriminant* $\Delta = F(\Sigma)$ for each of the following maps $F : \mathbf{R}^2 \to \mathbf{R}^2$:

(i) $F(x_1, x_2) = (x_1, x_1^2 + x_2^2)$

(ii) $F(x_1, x_2) = (x_1, x_1^3 - x_1 + x_2^2)$

(iii) $F(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2)$

(iv) $F(x_1, x_2) = (x_1, x_2^4 + x_1x_2^2)$**.**

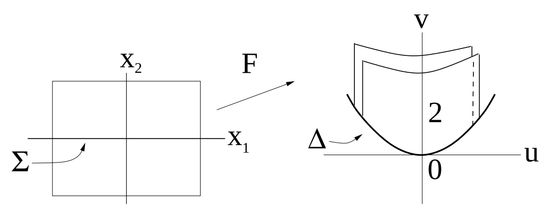
In each case label the various connected regions of the complement of Δ in $\mathbf{R}^2$ according to the number of points of $F^{-1}(q)$ for q in each region.

Answer

(i) $DF(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 2x_1 & 2x_2 \end{pmatrix}$**.**

det$= 2x_2$**,** $\Rightarrow \Sigma$ **is:** $x_2 = 0$**.**

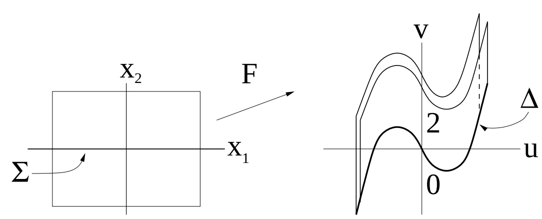
$F(x_1, 0) = (x_1, x_1^2) \Rightarrow \Delta$ **is:** $v = u^2$**.**



(ii) $DF(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 3x_1^2 - 1 & 2x_2 \end{pmatrix}$**.**

det$= 2x_2$**,** $\Rightarrow \Sigma$ **is:** $x_2 = 0$**.**

$F(x_1, 0) = (x_1, x_1^3 - 1) \Rightarrow \Delta$ **is:** $v = u^3 - u$**.**



(iii) $DF(x_1, x_2) = \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{pmatrix}$**.**

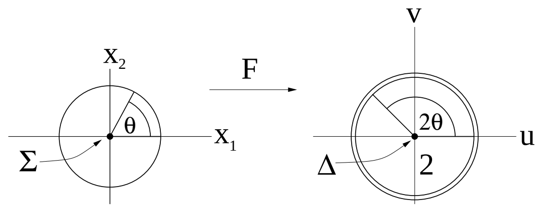
det$= 4(x_1^2 + x_2^2$**,** $\Rightarrow \Sigma$ **is** $(0, 0)$ **only.**

$\Delta = \{F(0, 0)\} = (0, 0)$ **only.**

In polar coordinates $(x_1, x_2) = r(\cos\theta, \sin\theta)$ **we see**

$$(u, v) = r^2) \cos 2\theta, \sin 2\theta)$$

i.e. $F(x) = x^2$ **in complex notation.**



Angles are doubled, radii are squared.

(iv) $DF(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ x_2^2 & 4x_2^3 + 2x_1x_2 \end{pmatrix}$**.**

$\det = 4x_2^3 + 2x_1x_2$**,** $\Rightarrow \Sigma$ **is:** $x_2 = 0$ **and** $x_1 = -2x_2^2$**.**

$F(x_1, 0) = (x_1, 0)$**,** $F(-2x_2^2, x_2) = (-2x_2^2, -x_2^4)$**, giving part of the parabola** $v = -\left(\frac{u}{2}\right)^2$ **with** $u \leq 0$**.**