Question

Use the Bromwich inversion integral to find the following:

(i) $L^{-1}\left[b \exp \frac{(-ap)}{(p^2+b^2)}\right], \ a > 0, \ t \neq a.$

What value do you get when $t = a$?

(ii) $L^{-1}[p^n], \ n =$positive integer.

(iii) $L^{-1}[p^{\frac{7}{2}}]$

(iv) $L^{-1}[p^{-\alpha}], \ a > 0.$

NB Consider al types of α.

(v) $L^{-1}[\exp(-p)\cosh(p)]$

(vi) $L^{-1}\left[\frac{\sqrt{p}}{(1-p^{\frac{3}{2}})}\right]$

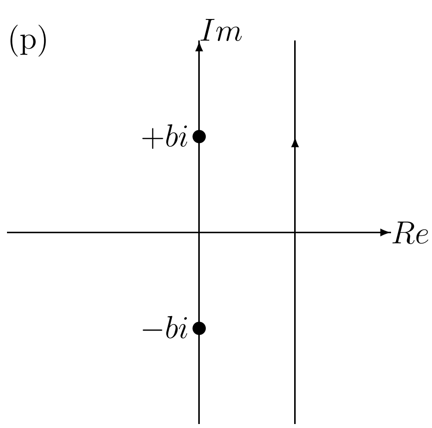
(The following results should be useful

Gamma function: $L[t^z] = \frac{z!}{p^{(z+1)}} = \frac{\Gamma(z+1)}{p^{(z+1)}}, \ \Gamma(-n) \to \infty, \ n = 0$ or positive integer

Delta function: $L[\delta(t-a)] = \exp(-pa), \ L[\delta(t)] = 1$)

Answer

(i) $f(t) = \frac{1}{2\pi i}\int dp \ \frac{de^{-ap}e^{pt}}{p^2+b^2}$



Simple poles at $\pm bi$.

First take $t > a$. Then you have $e^{p(t-a)}$ as a factor and you need to use a left hand semicircle (where $e^{p(t-a)}$ is exp. small) to complete the contour.

This now includes 2 poles with residues:

$$\begin{cases} \text{at p = ib} & \dfrac{be^{ib(t-a)}}{2ib} = \dfrac{e^{ib(t-a)}}{2i} \\ \text{at p = -ib} & \dfrac{be^{-ib(t-a)}}{-2ib} = \dfrac{-e^{-ib(t-a)}}{2i} \end{cases}$$

Hence $f(t) = \sin b(t-a)$ by residue calculus.

Now take $t > a$. Must complete with right hand semicircle. No poles included now, to $f(t) = 0$.

If $t = a$ need to evaluate $J = \displaystyle\int \frac{dp}{p^2+b^2} = f(a)$

Write it as:

$$\begin{aligned} J &= \frac{1}{2ib}\int\left\{\frac{1}{p-ib} - \frac{1}{p+ib}\right\}dp \\ &= \frac{1}{2ib}[\log(p-ib) - \log(p+ib)]_{c-i\infty}^{c+i\infty} \\ &= \frac{1}{2ib}\left[\log\left|\frac{p-ib}{p+ib}\right| + i\arg(p-ib) - i\arg(p+ib)\right]_{c-i\infty}^{c+i\infty} \\ &= 0 \end{aligned}$$

(since $\arg(p-ib)$ with $p = c + i\infty$ is $\dfrac{\pi}{2}$ etc $\cdots$)

so finally: $L^{-1}\left[\dfrac{be^{-ap}}{(p^2+b^2)}\right] = u(t-a)\sin b(t-a)$

(ii) Two methods:

(a) Use Gamma result that $L[t^z] = \dfrac{\Gamma(z+1)}{p^{(z+1)}}$

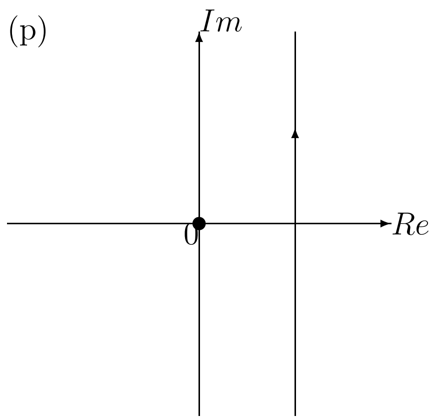
Then $\dfrac{t^z}{\Gamma(z+1)} = L^{-1}\left[p^{-(z+1)}\right]$ so putting $z+1 = -n$ we have

$L^{-1}[p^n] = \dfrac{t^{-n-1}}{\Gamma(-n)} = \dfrac{t^{-n-1}}{\infty} = 0$

(b) Use contour:

$$f(t) = \frac{1}{2\pi i} dp\ p^n e^{pt}$$

(p)



But no poles <u>anywhere</u> so if you complete to the <u>left</u> the closed contour encloses no poles. The integral therefore = 0. However we must check the semicircle contribution vanishes:

$$0 = \int + \int \Rightarrow \int = -\int$$

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This does in the left hand part of the plane, since $p^n e^{pt} \to 0$ as $p = Re^{i\theta}$ $R \to \infty$ $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ i.e., $Re(pt) < 0$ so $f(t) = 0$

This is a "non-sensical" result, since obviously $L[0] = 0$ and not p^n!!!Rather it's a $2 \to 1$ mapping. But if we <u>do</u> see p^n where n is a positive integer in a Laplace transform, we know it vanishes in the inverse transform.

(iii) $f(t) = \frac{1}{2\pi i} \int dp\ p^{+\frac{7}{2}} e^{pt}$

Here we can use the Gamma function result:
$L[t^z] = \frac{\Gamma(z+1)}{p^{(z+1)}}$ with $z + 1 = -\frac{7}{2} \Rightarrow z = -\frac{9}{2}$

Thus $L^{-1}[p^{\frac{7}{2}}] = \frac{t^{-\frac{9}{2}}}{\Gamma(-\frac{9}{2}+1)} = \frac{t^{-\frac{9}{2}}}{\Gamma(-\frac{7}{2})}$

What's $\Gamma(-\frac{7}{2})$? I's <u>not</u> zero (Γ is only zero when $\Gamma(-n)$ n integer > 0)

Well $\Gamma(z+1) = z\Gamma(z)$ (know this!)
(cf. factorial: $n! = n(n-1)!$)

Thus
$$\left.\begin{aligned} \left(-\frac{7}{2}\right)\Gamma\left(-\frac{7}{2}\right) &= \Gamma\left(-\frac{5}{2}\right) \\ \left(-\frac{5}{2}\right)\Gamma\left(-\frac{5}{2}\right) &= \Gamma\left(-\frac{3}{2}\right) \\ \left(-\frac{3}{2}\right)\Gamma\left(-\frac{3}{2}\right) &= \Gamma\left(-\frac{1}{2}\right) \\ \left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right) &= \Gamma\left(\frac{1}{2}\right) \end{aligned}\right\}$$

$$\Rightarrow \Gamma\left(-\frac{7}{2}\right) = \left(-\frac{2}{7}\right)\left(-\frac{2}{5}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{1}\right)\Gamma\left(\frac{1}{2}\right) = \frac{16}{105} \times \sqrt{\pi}$$

<u>Spin off result</u>:
Contour integral method shows that by completing to the right,

$$\frac{t^{-\frac{9}{2}}}{\Gamma\left(-\frac{7}{2}\right)} = \frac{1}{2\pi i} \int_{PICTURE} dp\ p^{\frac{7}{2}} e^{pt}$$

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Hence a contour integral representation for $\frac{1}{\Gamma(-z)}$ is

$$\frac{1}{\Gamma(-z)} = \frac{1}{2\pi i} \int_{PICTURE} dp\ p^z e^p$$

(iv) $L^{-1}[p^{-\alpha}]\ \alpha > 0$

$$f(t) = \frac{1}{2\pi i} \int \frac{e^{pt}}{p^\alpha} dp$$

If α were an integer $\Rightarrow$ pole of order n at $p = 0$, complete by a left semicircle where $Re(pt) < 0$. The residue gives $f(t) = \frac{t^{n-1}}{(n-1)!}$ (from standard residue of higher order pole formula)

If $0 < \alpha < 1$ move the integral to the contour

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The small circle around zero gives zero as it shrinks leaving

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \left[\int_{\infty e^{-i\pi}}^{0} + \int_{0}^{\infty e^{+i\pi}} \right] \frac{e^{pt}}{p^\alpha} dp \\ &= \frac{1}{2\pi i} \int_0^\infty \frac{e^{-xt}}{x^\alpha} dx \left[e^{+i\alpha\pi} - e^{-i\alpha\pi} \right] \\ &= \frac{1}{2\pi} \alpha! 2i \sin \alpha\pi \\ &= \frac{\alpha! \sin \alpha\pi}{\pi} \\ &= \frac{\Gamma(\alpha+1) \sin \alpha\pi}{\pi} \end{aligned}$$

Result.

Spin off: Thus we deduce from (iii) that the identity

$$\frac{1}{\Gamma(-z)} = \Gamma(z+1)\frac{\sin z\pi}{\pi} \Rightarrow \underline{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}}$$

The "reflection formula" for the Γ-reduction.

(v) $L^{-1}\left[e^{-p}\cosh p\right]$

$$f(t) = \frac{1}{2\pi i}\int_{\uparrow} e^{-p}\cosh p e^{pt} dp = \frac{1}{2\pi i}\int_{\uparrow} e^{p(t-1)}\cosh p \; dp$$

$$= \frac{1}{2\pi i}\int_{\uparrow} e^{p(t-1)}\left(\frac{e^{p}+e^{-p}}{2}\right) = \frac{1}{4\pi i}\int_{\uparrow} e^{pt} dp + \frac{1}{4\pi i}\int_{\uparrow} e^{p(t-2)}$$

Now use δ-function result:

$$L[\delta(t-a)] = e^{-pa}$$

$\Rightarrow$

$$L^{-1}\left[e^{-p}\cosh p\right] = \frac{1}{2}[\delta(t=0)] + \frac{1}{2}[\delta(t-2)] = \frac{1}{2}\delta(t) + \frac{1}{2}\delta(t-2)$$

This result follows from the contour integral method by deforming contours to complete where $Re(pt)$ and $Re(p(t-2)) < 0$ respectively. The integrals enclose no singularities for $t \neq 0, \; t \neq 2$ and the semicircle contributions vanish.

Thus the $\int_{\uparrow} e^{pt} dp$ and $\int_{\uparrow} e^{p(t-2)} dp$ both $= 0$

unless $t = 0$ or $t = 2$.

When $t = 0$ or $t = 2$ they don't converge [$\int_{\uparrow} dp$ is infinite]

so we have δ-function spikes there.