THEORY OF NUMBERS
TCHEBYCHEV'S THEOREM

Prime numbers We denote $\prod(x) = \sum_{p \le x} 1$. So $\prod(p_n) = n \;\; \prod(x) < x$.

Theorem $\sum_p \frac{1}{p}$ diverges.

Proof $\prod_{p \le N} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \le N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) = \sum_{n=1}^{\infty} \frac{\theta_n}{n}$

where $\theta_N = \begin{cases} 1 & \text{if every prime factor of } n \text{ is } \le N \\ 0 & \text{otherwise} \end{cases}$

$\ge \sum_{n=1}^{N} \frac{1}{n} \ge \log N$

$\log N \le \prod_{p \le N} \left(1 - \frac{1}{p}\right)^{-1}$

Therefore

$$\begin{aligned} \log \log N &\le \sum_{p \le N} - \log \left(1 - \frac{1}{p}\right) \\ &= \sum_{p \le N} \frac{1}{p} + \sum_{p \le N} \left(\frac{1}{p^2} + \frac{1}{p^3} + \dots\right) \\ &= \sum_{p \le N} \frac{1}{p} + C \end{aligned}$$

Therefore $\sum_{p \le N} \frac{1}{p} > \log \log N - C$

Tchebychev's Theorem $\exists$ positive constants A, B such that $A \frac{x}{\log x} < \prod(x) < B \frac{x}{\log x}$ for large x.

Proof

Definition

$$\begin{aligned} \Lambda(n) &= \begin{cases} \log p & \text{if } n = p^l \\ 0 & \text{otherwise} \end{cases} \\ \Psi(x) &= \sum_{n \le x} \Lambda(n) \\ \theta(x) &= \sum_{p \le x} \log p \\ F(x) &= \sum_{n \le x} \log n \end{aligned}$$

Lemma A $F(x) = x \log x - x + O(\log x)$ $x \geq 2$

Corollary $F(x) - 2F\left(\frac{1}{2}x\right) = x \log 2 + O(\log x)$ $x \geq 2$

Proof $F(x) = \sum_{2 \leq n \leq x} \log n$

Now $\log t$ increases as t increases and so

$$\int_{n-1}^{n} \log t \, dt \leq \log n \leq \int_{n}^{n+1} \log t \, dt n \geq 2$$

So

$$\int_{1}^{[x]} \log t \, dt \leq F(x) \leq \int_{1}^{[x]+1} \log t \, dt$$

$$[t \log t - t]_1^{[x]} \leq F9x) \leq [t \log t - t]_1^{[x]+1}$$

Where $F(x) = x \log x - x + O(\log x)$

Lemma B $\Psi(x) - \Psi\left(\frac{1}{2}x\right) \leq F(x) - 2F\left(\frac{1}{2}x\right) \leq \psi(x)$ $x \geq 2$

Proof $\log n = \sum_{d|n} \Lambda(d)$ $n = 1, 2, \ldots$

$$\begin{aligned} F(x) &= \sum_{n \leq x} \log n = \sum_{n \leq x} \sum_{d|n} \Lambda(d) \\ &= \sum_{d \leq x} \Lambda(d) \sum_{n \;\; d|n \leq x} 1 \\ &= \sum_{d \leq x} \lambda(d) \sum_{m \;\; m \leq \frac{x}{d}} 1 \\ &= \sum_{d \leq x} \Lambda(d) \left[\frac{x}{d}\right] \end{aligned}$$

$$\begin{aligned} F(x) - 2F\left(\frac{1}{2}x\right) &= \sum_{d \leq x} \Lambda(d) \left[\frac{x}{a}\right] - 2 \sum_{d \leq \frac{1}{2}x} \Lambda(d) \left[\frac{\frac{1}{2}x}{d}\right] \\ &= \sum_{\frac{1}{2}x < d \leq x} \Lambda(d) \left[\frac{x}{d}\right] + \sum_{d \leq \frac{1}{2}x} \Lambda(d) \left\{\left[\frac{x}{d}\right] - 2\left[\frac{\frac{1}{2}x}{d}\right]\right\} \\ &= \sum_1 + \sum_2 \end{aligned}$$

$\sum_1 = \sum \Lambda(d) = \Psi(x) - \Psi\left(\frac{1}{2}x\right)$

Now $f(\alpha) = [\alpha] - 2\left[\frac{1}{2}\alpha\right]$ is periodic with period 2.

$f(\alpha) = \begin{cases} 0 & \text{if } 0 \le \alpha < 1 \\ 1 & \text{if } 1 \le \alpha < 2 \end{cases}$

Therefore $0 \le f(\alpha) \le 1$ for all real α

therefore $0 \le \sum_2 \le \sum d \le \frac{1}{2}x\Lambda(d) = \Psi\left(\frac{1}{2}x\right)$

Hence the result.

Lemma C **(i)** $\Psi(x) \ge x\log 2 + O(\log x)$

(ii) $\Psi(x) \le 2x\log 2 + O(\log^2 x)$

Proof **(i)** Immediate by B and corollary A.

(ii) Choose l to satisfy $1 \le x2^{-l} < 2$

Then

$$\begin{aligned}
\Psi(x) &= \Psi(x) - \Psi(x2^{-l}) \\
&= \sum_{n=0}^{l-1}\left\{\Psi(x2^{-n}) - \Psi(x2^{-n-1})\right\} \\
&\le x\log 2\sum_{n=0}^{l-1} + O(l\log x) \text{ by } B \\
&\le 2x\log 2 + O(\log^2 x)
\end{aligned}$$

Lemma D $\theta(x) = \Psi(x) + O\left\{(x^{\frac{1}{2}})\log^2 x\right\}$

Proof

$$\begin{aligned}
0 &\le \Psi(x) - \theta(x) \\
&= \sum_{p,l(p^l \le x,\ l \ge 2)} \log p \\
&\le \frac{\log x}{\log 2}.x^{\frac{1}{2}}\log x
\end{aligned}$$

Corollary $\theta(x) = O(x)$

Lemma E

$$\begin{aligned}
\prod(x) &= \frac{\theta(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right) \\
&= \frac{\Psi(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right)
\end{aligned}$$

Proof Suppose $x \ge 3$

$$
\begin{aligned}
\prod(x) &= \sum_{p\le x} 1 \\
&= \sum_{2\le n\le x} \frac{\theta(n)-\theta(n-1)}{\log n} \\
&= \frac{\theta([x])}{\log([x])} + \sum_{2\le n\le x-1} \theta(n)\left(\frac{1}{\log n} - \frac{1}{\log(n-1)}\right) \\
&= \frac{\theta([x])}{\log([x])} + O \sum_{2\le n\le x-1} \frac{1}{\log^2 n} \\
&= \frac{\theta(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right)
\end{aligned}
$$

The theorem follows from Lemmas E, D, C if we take

$$A < \log 2, \;\; B > 2\log 2.$$

It follows that $\exists K_1, k_2$ such that

$$k_1 m \log n < p_n < k_2 n \log n.$$