Question

Repairs carried out in a workshop often involve the replacement of a particular component. In the interests of efficiency a small stock of components is kept in the workshop. The probability that k components are used in the workshop during a working day is a_k where

$$\begin{array}{ll} a_k > 0 & k = 0, 1, 2, ..., 12, \\ a_k = 0 & \text{otherwise.} \end{array}$$

and the numbers of components used on different days are independent, identically distributed random variables.

At the end of each working day the number of components in stock is counted. If there are fewer than 12 components, then the minimum number of boxes, each containing 5 components, are brought from the central stores to make the stock in the workshop at least 12 components.

Show that the number of components if necessary, form a Markov chain. Show that the Markov chain is ergodic and find the equilibrium distribution.

Answer

Let X_n = number of customers in the workshop at the end of day n - after re stocking.

Let Z_n = the number of components used during day n.

X_n depends only on X_{n-1} and Z_n, so the history of the process before the end of day $(n-1)$ is irrelevant.

The possible values of X_n are $12, 13, 14, 15, 16$ and in terms of N_{n-1} and Z_n are given by the table below, with transition probabilities as shown.

Z_n / X_{n-1}	0	1	2	3	4	5	6	7	8	9	10	11	12
12	12	16	15	14	13	12	16	15	14	13	12	16	15
13	13	12	16	15	14	13	12	16	15	14	13	12	16
14	14	13	12	16	15	14	13	12	16	15	14	13	12
15	15	14	13	12	16	15	14	13	12	16	15	14	13
16	16	15	14	13	12	16	15	14	13	12	16	15	14

$$
\begin{array}{rcll}
P_{xx} & = & P(Z_n = 0) + P(Z_n = 5) & +P(Z_n = 10) \\
P_{xx} & = & a_0 + a_5 + a_{10} & x = 12, \ldots, 16 \\
P_{x,x+1} & = & a_4 + a_9 & x = 12, \ldots, 15 \\
P_{x,x-1} & = & a_1 + a_6 + a_{11} & x = 13, \ldots, 16 \\
P_{x,x+2} & = & a_3 + a_8 & x = 12, 13, 14 \\
P_{x,x-2} & = & a_2 + a_7 + a_{12} & x = 14, 15, 16 \\
P_{x,x+3} & = & a_2 + a_7 + a_{12} & x = 12, 13 \\
P_{x,x-3} & = & a_3 + a_8 & x = 15, 16 \\
P_{x,x+4} & = & a_1 + a_6 + a_{11} & x = 12 \\
P_{x,x-4} & = & a_4 + a_9 & x = 16
\end{array}
$$

$$
\begin{array}{rcl}
\text{Let } p_0 & = & a_0 + a_5 + a_{10} \\
p_1 & = & a_4 + a_9 \\
p_2 & = & a_3 + a_8 \\
p_3 & = & a_2 + a_7 + a_{12} \\
p_4 & = & a_1 + a_6 + a_{11}
\end{array}
$$

The transition matrix is

$$
p = \begin{array}{c} 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{array}
\begin{pmatrix}
p_0 & p_1 & p_2 & p_3 & p_4 \\
p_4 & p_0 & p_1 & p_2 & p_3 \\
p_3 & p_4 & p_0 & p_1 & p_2 \\
p_2 & p_3 & p_4 & p_0 & p_1 \\
p_1 & p_2 & p_3 & p_4 & p_0
\end{pmatrix}
$$

All the p_i are positive (non zero) and so the Markov chain is aperiodic irreducible and finite, and therefore ergodic. Hence the equilibrium distribution is also the stationary distribution, satisfying: $\pi P = \pi$
Now $\pi = (1, 1, 1, 1, 1)$ satisfies this, since the column sums for P are all 1 (as well as the row sums), so the required probability distribution is

$$
\pi = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right)
$$