Question

Classify the states of the infinite Markov chain with probability transformation matrix P as transient, null-recurrent or positive-recurrent, and periodic or aperiodic where

$$P = \begin{pmatrix} p & 0 & 1-p & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1-p & 0 & p & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & p & 1-p & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & p & 0 & 1-p & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & p & 0 & 0 & 1-p & 0 & \cdot \\ \cdot & \cdot & 0 & p & 0 & 0 & 0 & 1-p & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Answer

(i) $p = 1$: States $1, 2, 3, 4$ are all absorbing states. States $5, 6, 7, 8, \ldots$ are ephemeral, i.e. $p_{jj} = 0, \quad j = 5, 6, \ldots$

(ii) $p = 0$: States $\{1, 3\}$ form a 2-state Markov chain with transition matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so both states are periodic with period 2 and are positive recurrent with mean recurrence time 2.

State 2 is absorbing.

States $4, 5, 6, \ldots$ are transient ($p_{jj} = 0$). Infact the Markov chain follows the route $4 \to 5 \to 6 \to 7 \to 8 \ldots$ with probability 1.

(iii) $0 < p < 1$

States $\{1, 3\}$ form a 2-state Markov chain, both states aperiodic. From general results in lectures for a 2×2 Markov chain both states are positive recurrent with $\mu_1 = 2 = \mu_3$.

State 2 is absorbing.

$\{4, 5, 6, ...\}$ is a closed irreducible set of states. The first return to state 4 in n steps follows only the path $4 \to 5 \to 6 \to \ldots (n+2) \to (n+3) \to 4$.

So P(1st return to 4 at step n) $= f_{44}^{(n)} = p(1-p)^{n-1}$

So $f_{44} = p\sum_{n=1}^{\infty}(1-p)^{n-1} = 1 : \quad \mu_4 = p\sum_{n=1}^{\infty} n(1-p)^{n-1} = \dfrac{1}{p}$

So state 4 is positive recurrent and aperiodic. State 4 intercommunicates with $5, 6, 7, \ldots$ so they are all positive recurrent and aperiodic.

In fact it can be seen directly from the matrix that states $5, 6, 7, \ldots$ will all be similar to state 4 in their behaviour.