Question

Define the terms equilibrium distribution and stationary distribution for a Markov chain. Explain how they are related for a finite Markov chain.

Consider the following experiment. Initially 6 fair coins are tossed and X_0 is the total number of heads obtained. One coin is then selected at random and turned over and X_1 is the total number of heads now showing. A coin is again selected at random and turned over giving X_2 heads, and so on.

Discuss briefly why $\{X_k\}, \quad k = 0, 1, 2, ...,$ forms a Markov chain on the states $0, 1, 2, \ldots, 6$. Write down the initial probabilities of occupying the states and the transition probability matrix.

Obtain the stationary distribution of the Markov chain. Hence find the probability distribution of $X_k \quad (k = 1, 2, 3, \ldots)$.

If the number of heads showing initially is known to be 2, calculate the probability distribution for the number of heads showing after 2 coins have been turned over.

Answer

A Markov chain with transition matrix P has an equilibrium distribution if $\mathbf{p}^{(n)} = \mathbf{p}^0 P^n \to \boldsymbol{\pi}$ as $n \to \infty$, independently of the initial distribution $\mathbf{p}^{(0)}$.

$\boldsymbol{\pi}^*$ is a stationary distribution if $\boldsymbol{\pi}^* P = \boldsymbol{\pi}^*$.

If $\boldsymbol{\pi}$ is an equilibrium distribution then it it is a stationary distribution, but not conversely. An irreducible finite Markov chain with aperiodic states has a unique stationary distribution which is also its equilibrium distribution.

X_n depends only on X_{n-1} i.e. how many heads there are at that stage, and not how X_{n-1} has been arrived at. The initial probabilities are

$$p_0 = p_6 = \frac{1}{64} \quad p_1 = p_5 = \frac{6}{64} \quad p_2 = p_4 = \frac{15}{64} \quad p_3 = \frac{20}{64} \text{ -- binomial}$$

The transition matrix is:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{6} & 0 & \frac{4}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{6} & 0 & \frac{3}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{6} & 0 & \frac{2}{6} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Note: The Markov chain is irreducible. All states are positive recurrent. All are periodic, with period 2.

Suppose the stationary distribution is $(\pi_0, ..., \pi_6)$. Solving $\boldsymbol{\pi} = \boldsymbol{\pi} P$ gives:
$\boldsymbol{\pi}^* = \left(\frac{1}{64}, \frac{6}{64}, \frac{15}{64}, \frac{20}{64}, \frac{15}{64}, \frac{6}{64}, \frac{1}{64}\right)$

$$
\begin{aligned}
\text{If} \quad \mathbf{p}^0 &= (0,\, 0,\, 1,\, 0,\, 0,\, 0,\, 0) \\
\mathbf{p}^{(1)} &= (0,\, \tfrac{2}{6},\, 0,\, \tfrac{4}{6},\, 0,\, 0,\, 0) \\
\mathbf{p}^{(2)} &= (\tfrac{2}{36},\, 0,\, \tfrac{22}{36},\, 0,\, \tfrac{12}{36},\, 0,\, 0)
\end{aligned}
$$

There is no equilibrium distribution since we have periodicity.
The vector of initial probabilities is the same as $\boldsymbol{\pi}^*$. Thus the probability distribution of X_k is $\boldsymbol{\pi}^* P^k = \boldsymbol{\pi}^*$.