Question

An electrician services m machines which experiences random breakdowns. If a machine is working at time t then the probability that it will require attention in the time interval $(t, t + \delta t]$ is $\lambda \delta t + o(\delta t)$ for each machine. The machines work independently and the electrician can only service one machine at a time. The service times are independent random variables with cumulative distribution function $1 - e^{-\mu x}$ (for $x \geq 0$). Let $X(t)$ denote the number of machines working at time t.
Show that the limiting probabilities

$$P_j = \lim_{t \to \infty} p\{X(t) = j\} \quad j = 0, 1, 2, \ldots, m$$

satisfy the equations

$$j\lambda p_j - \mu p_{j-1} = (j+1)\lambda p_{j+1} - \mu p_j \quad (j = 0, 1, 2, \ldots, m-1),$$

where $p_{-1} = 0$ and $m\lambda p_m = \mu p_{m-1}$.
By deducing that $j\lambda p_j - \mu p_{j-1} = 0$, or otherwise, show that in the long run the expected number of machines working simultaneously is $(1 - p_m)\frac{\mu}{\lambda}$.

Answer

Service times have c.d.f. $1 - e^{-\mu x}$ i.e. negative exponential. So they are competed according to a Poisson process with rate μ.
When $X(t) = j$ the change in $(t, t + \delta t]$ is

$$\begin{array}{rll} +1 & \text{with probability} & \mu \delta t + o(\delta t) \\ -1 & \text{with probability} & \lambda j \delta t + o(\delta t) \\ 0 & \text{with probability} & 1 - (\mu + \lambda j)\, \delta t + o(\delta t) \end{array}$$

When $X(t) = 0$ the change in $(t, t + \delta t]$ is

$$\begin{array}{rll} +1 & \text{with probability} & \mu \delta t + o(\delta t) \\ 0 & \text{with probability} & 1 - \mu \delta t + o(\delta t) \end{array}$$

When $X(t) = m$ the change in $(t, t + \delta t]$ is

$$\begin{array}{rll} -1 & \text{with probability} & \lambda m \delta t + o(\delta t) \\ 0 & \text{with probability} & 1 - \lambda m \delta t + o(\delta t) \end{array}$$

$$\begin{aligned}
P(X(t+\delta t)=j) &= P(X(t+\delta t)=j|X(t)=j-1)P(X(t)=j-1) \\
&+ P(X(t+\delta t)=j|X(t)=j+1)P(X(t)=j+1) \\
&+ P(X(t+\delta t)=j|X(t)=j)P(X(t)=j)
\end{aligned}$$

$$\begin{aligned}
p_j(t+\delta t) &= (\mu\delta t+o(\delta t))p_{j-1}(t) \\
&+ (\lambda(j+1)+o(\delta t))p_{j+1}(t) \\
&+ (1-(\lambda j+\mu)\delta t+o(\delta t))p_j(t) \quad (0<j<m)
\end{aligned}$$

giving

$$p'_j(t)=\mu p_{j-1}(t)+\lambda(j+1)p_{j+1}(t)-(\lambda j+\mu)p_j(t)$$

Similarly

$$\begin{aligned}
p'_0(t) &= -\mu p_0(t)+\lambda p_1(t) \\
p'_m(t) &= -\lambda m p_m(t)+\mu p_{m-1}(t)
\end{aligned}$$

The equilibrium equations are therefore:

$$\begin{aligned}
0 &= -\mu p_0+\lambda p_1 \\
0 &= -\lambda m p_m+\mu p_{m-1} \\
0 &= \mu p_{j-1}+\lambda(j+1)p_{j+1}-(\lambda j+\mu)p_j \quad (0<j<m)
\end{aligned}$$

So $j\lambda p_j-\mu p_{j-1}=(j+1)\lambda p_{j+1}-\mu p_j \quad (0<j<m)$
Since $1\cdot\lambda p_1-\mu p_0=0$ it follows by induction that $j\lambda p_j-\mu p_{j-1}=0$ for $0<j<m$. Also $m\lambda p_m-\mu p_{m-1}=0$.
The expected number of machines working is

$$\sum_{j=1}^{m} jp_j=\frac{\mu}{\lambda}\sum_{j=1}^{m} p_{j-1}=\frac{\mu}{\lambda}(1-p_m)$$