Question

A man with initial capital £z gambles at a casino which may be assumed infinitely rich. He plays a series of independent games and has probabilities p of winning £1 and $q = 1-p$ of losing his £1 stake. Write down a difference equation and boundary condition for the probability q_z of his eventual ruin. Discuss briefly why these are insufficient for determining q_z.

Consider the following strategy for the gambler. On the first day he bets each of his pounds of capital in turn in z games and puts winnings and retained stake money, totalling £X_1, in a kitty. On the second day he bets with each of the X_1 pounds in the kitty and puts winnings and retained stake money, totalling £X_2, in another kitty, and so on Show that $\{X_n\}$ $n = 1, 2, \ldots$ is a branching chain with a probability of ultimate extinction given by

$$q_z = \begin{cases} \left(\frac{q}{p}\right)^z & \text{if } p > q \\ 1 & \text{if } p \leq q \end{cases}$$

Answer

$$\begin{aligned} q_z &= pq_{z+1} + qq_{z-1} \quad z = 1, 2, 3, \ldots \\ q_0 &= 1 \end{aligned}$$

The general solution is

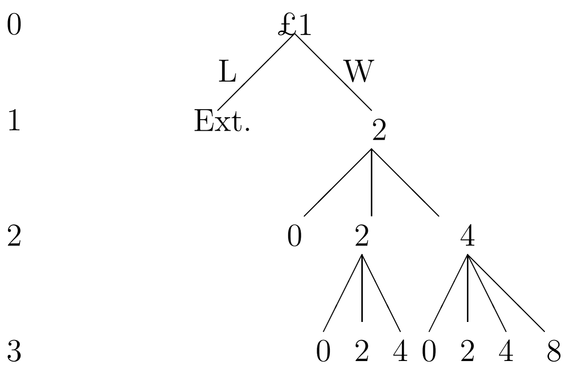
$$q_z = \begin{cases} A + B\left(\dfrac{q}{p}\right)^z & p \neq q \\ A + Bz & p = q \end{cases}$$

Th obtain a particular solution we need two boundary conditions, whereas we only have one. Note that because $0 \leq q_z \leq 1$, when $p = q$ this implies $B = 0$ and therefore $A = 1 = q_0$.

Also when $p < q$ we must have $B = 0$ and therefore $A = 1 = q_0$. So it is only when $p > q$ that we have insufficient information.

Consider a particular £1 and the situation on subsequent days.

End of day



etc

So we have a branching chain. Each starting £1 acts independently. The probabilities for the "offspring "of any individual £1 are

$$\begin{aligned} P(Z=0) &= q \\ P(Z=2) &= p \\ P(Z=k) &= 0 \text{ otherwise} \end{aligned}$$

so the p.g.f. is

$$A(s) = q + ps^2.$$

The Fundamental Theorem for branching chains says that the probability of ultimate extinction is the smallest positive root of the equation $s = A(s)$

$$\text{So} \;\; s = q + ps^2 \quad ps^2 - s + q = 0$$

$$i.e. \; (ps-q)(s-1) = 0 \;\; (p+q=1)$$

So the roots are

$$s = 1 \;\; s = \frac{q}{p}$$

So the probability of extinction with £1 is $\frac{q}{p}$ if $q < p$ or 1 if $q \geq p$.

Starting with £z, by independence the probability of extinction is

$$1 \text{ if } \;\; q \geq p; \;\; \left(\frac{q}{p}\right)^z \;\; \text{if} \;\; q < p$$