Question

What is meant by a "closed set of states" of a Markov chain? Explain the use of closed sets in classifying states as

(i) positive-recurrent,null-recurrent or transient,

(ii) period or aperiodic,

giving definitions of these terms.

The following transition matrix is a Markov chain with states $1, 2, 3, \ldots, 10$.

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \end{pmatrix}$$

Use a transition diagram to identify the closed sets of states and write the matrix in blocked form.

Carry out the above classification of the states. For each positive recurrent state calculate the mean recurrence time.

Answer

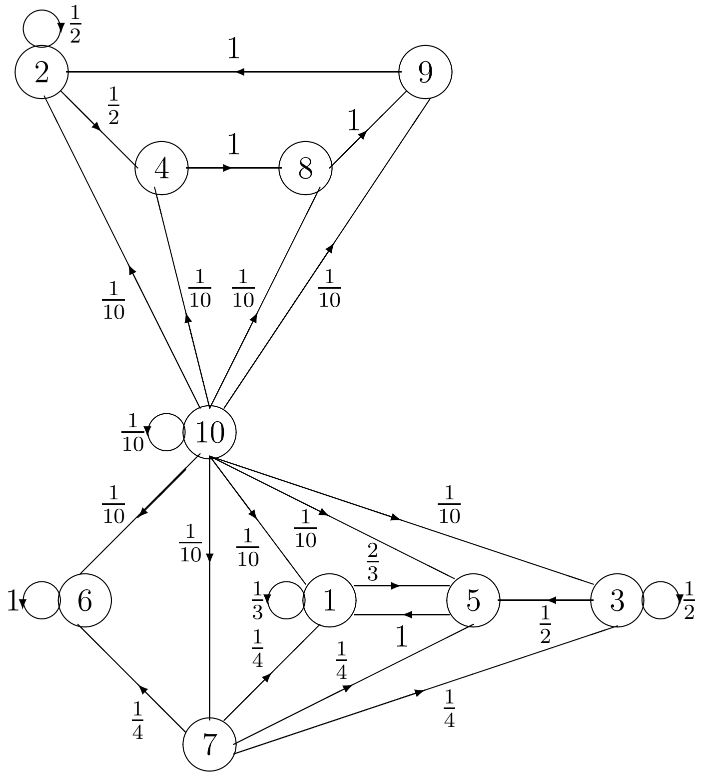
A set of states of a Markov chain is closed if no state outside C can be reached from a state in C. We can treat each closed set as a sub-chain in classifying states.

In classifying states we use the following results

1. If two states intercommunicate, they are of the same type.

2. The states space of a Markov chain decomposes into disjoint sets

- **(i)** T - consisting of all transient states
- **(ii)** $C_1, C_2, \ldots$ irreducible closed sets of recurrent states, therefore of the same type.



	3	7	10		1	5		2	4	8	9		6
3	$\frac{1}{2}$	0	0	⋮	0	$\frac{1}{2}$	⋮	0	0	0	0	⋮	0
7	$\frac{1}{4}$	0	0	⋮	$\frac{1}{4}$	$\frac{1}{4}$	⋮	0	0	0	0	⋮	$\frac{1}{4}$
10	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	⋮	$\frac{1}{10}$	$\frac{1}{10}$	⋮	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	⋮	$\frac{1}{10}$
	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
1	0	0	0	⋮	$\frac{1}{3}$	$\frac{2}{3}$	⋮	0	0	0	0	⋮	0
5	0	0	0	⋮	1	0	⋮	0	0	0	0	⋮	0
	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
2	0	0	0	⋮	0	0	⋮	$\frac{1}{2}$	$\frac{1}{2}$	0	0	⋮	0
4	0	0	0	⋮	0	0	⋮	0	0	1	0	⋮	0
8	0	0	0	⋮	0	0	⋮	0	0	0	1	⋮	0
9	0	0	0	⋮	0	0	⋮	1	0	0	0	⋮	0
	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
6	0	0	0	⋮	0	0	⋮	0	0	0	0	⋮	1

States 3, 7, 10 are transient.

For states $\{1,5\}$, $f_{11} = \frac{1}{3} + \frac{2}{3} \cdot 1 = 1$ so they are both recurrent.
Since the closed set is finite they are both positive recurrent.

$$\begin{aligned} \mu_1 &= \frac{1}{3} \times 1 + \frac{2}{3} \times 2 = \frac{5}{3} \\ \mu_5 &= \sum_{r=2}^{\infty} r \frac{2}{3} \left(\frac{1}{3}\right)^{r-2} = \frac{5}{2} \end{aligned}$$

Note that μ_5 could be calculated from the relation $\frac{3}{5} + \frac{2}{5} = 1$.
For states $\{2,4,8,9\}$ $f_{22} = \frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 \cdot 1 = 1$ so all are positive recurrent.

$$\begin{aligned} \mu_2 &= \frac{1}{2} \times 1 + \frac{1}{2} \times 4 = \frac{5}{2} \\ \mu_4 &= \sum_{r=4}^{\infty} r \left(\frac{1}{2}\right)^{r-4} \cdot \frac{1}{2} = 5 (= \mu_8 = \mu_9) \end{aligned}$$

State 6 is absorbing ($\mu_6 = 1$).
All states are aperiodic.