Question

A gambler with capital £z bets against an opponent with capital £$(a-z)$ using £1 stakes. On each bet the gambler wins £1 with probability $\frac{1}{3}$, loses £1 with probability $\frac{1}{2}$ and retains his stake in the event of a draw. Show that the probability, P_z, that the game ends after an odd number of bets satisfies the second order difference equation

$$2P_{z+1} + 7P_z + 3P_{z-1} = 6 \quad \text{for} \quad z = 1, 2, ..., a-1.$$

By solving the equation, or otherwise, calculate the probability that the game will end after an odd number of bets when the gambler and his opponent both start with £2.
Investigate the value of P_z if the gambler plays against an infinitely rich opponent.

Answer

The Gambler has £ z. The opponent has £$(z-a)$ pounds.
P_z =Prob(game ends after as odd number of bets)
Consider the first bet. There are 3 outcomes.

(i) gambler wins. Then he has £$(z+1)$ and the game must not end in a further odd number of bets. Probability $\frac{1}{3}(1-P_{z+1})$

(ii) gambler loses. Then he has £$(z-1)$ and the games must not end in a further odd number of bets. Probability $\frac{1}{2}(1-P_{z-1})$

(iii) draw. Then the gambler still has £ z and the game mustn't end in an odd number of bets. Probability $\frac{1}{6}(1-P_z)$

So $P_z = \frac{1}{3}(1-P_{z+1}) + \frac{1}{2}(1-P_{z-1}) + \frac{1}{6}(1-P_z)$ for $0 < z < a$. Also $P_0 = 0$ and $P_a = 0$
The equation when simplified becomes

$$2P_{z+1} + 7P_z + 3P_{z-1} = 6 \quad 0 < z < a$$

Putting $P_z = \lambda^z$ in the homogeneous equation gives

$$2\lambda^2 + 7\lambda + 3 = 0 \Rightarrow (2\lambda+1)(\lambda+3) = 0 \Rightarrow \lambda = -\frac{1}{2}, -3$$

A particular solution of the inhomogeneous equation is $P_z = \frac{1}{2}$ (constant).
So the general solution is

$$P_z = A\left(-\frac{1}{2}\right)^z + B(-3)^z + \frac{1}{2}$$

$P_0 = 0$ so $A + B + \dfrac{1}{2} = 0$

$P_a = 0$ so $A\left(-\dfrac{1}{2}\right)^a + B(-3)^a + \dfrac{1}{2} = 0$

Solving gives $B = \dfrac{1-(-2)^a}{2(6^a - 1)} \Rightarrow A = -B - \dfrac{1}{2}$

So $P_z = \dfrac{1-(-2)^a}{2(6^a - 1)}\left[(-3)^z - \left(-\dfrac{1}{2}\right)^z\right] + \dfrac{1}{2}\left[1 - \left(-\dfrac{1}{2}\right)^z\right]$

When $z = 2$ and $a = 4$, $P_z = \dfrac{1-2^4}{2(6^4-1)}\left(9 - \dfrac{1}{4}\right) + \dfrac{1}{2}\left(1 - \dfrac{1}{4}\right) = \dfrac{12}{37} = 0.\overline{324}$

In general, when $a \to \infty$, $\quad P_z = \dfrac{1}{2}\left(1 - \left(-\dfrac{1}{2}\right)^z\right)$

So $P_z \to \dfrac{3}{8}$